Two applications of bigness

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These are some cursory notes for my pair of talks in the Berkeley number theory student seminar. They are meant to give an exposition of the recent adelic line bundle theory of Yuan–Zhang, focusing less on details and more on its Diophantine applications. All errors and pedantry are due to me—please send me any comments or corrections!

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1 Introduction

The goal of these notes/talks is to illustrate some recent results using the framework of *adelic line bundles over quasiprojective varieties* due to Yuan–Zhang [YZ24]. The key theme in these results is the *bigness* of some line bundle in question and the application of this towards fundamental properties of heights. Since the framework is rather technical with a good amount of data to keep track of, we will have to be rather sketchy with the details in order to get the main ideas across in time. Many of the footnotes try to point out where this sketchiness is taking place.

In these notes, a *curve* will be a geometrically connected and projective reduced[1] scheme of pure dimension 1, unless otherwise specified. A *variety* will be an *integral* separated scheme of finite type over a field (sometimes I might slip up and call something a variety when it is not necessarily integral, but I hope that won't happen). All schemes will be Noetherian. By a global field K we mean either a number field or a function field of one variable over a field k, the latter by which we mean the function field of a smooth projective curve over k. We write the group operation on the Picard group additively, so nL for a line bundle L and $n \in \mathbf{Z}$ means $L^{\otimes n}$. By $|D|$ for a divisor D, we mean the support of D. Base changes via tensor/fiber products are denoted with the appropriate subscript; for instance, if A is an abelian group, then $A_{\mathbf{Q}}$ means $A \otimes_{\mathbf{Z}} \mathbf{Q}$. All of the other notation should be standard.

1.1 Main theorems

We will state the two results we would like to discuss. The first is due to Yuan:

Theorem 1.1.1. [Yua24, Theorem 1.4, function field case] Let K be a number field or a function field of one variable over a field k. Let C be a curve of genus $g > 1$ over K such that $C_{\overline{K}}$ is non-isotrivial over $\overline{k}$ if we are in the function field case[2]. Then there are constants

[1] Not necessarily integral, as in the case of (semi)stable curves which we will need below.

[2] Here, non-isotrivial means that $C_{\overline{K}}$ is not the base change of a curve defined over $\overline{k}$, so that the family of curves defined by $C_{\overline{K}}$ is a genuinely varying family.

$c_1, c_2 > 0$, depending only on g, such that

$$c_1 \max(h_{Fal}(C), 1) \leq [\overline{\omega}^2_{C/K,a}] \leq c_2 \max(h_{Fal}(C), 1).$$

Here $[\overline{\omega}^2_{C/K,a}]$ is $\overline{\omega}^2_{C/K,a}/[K : \mathbf{Q}]$ if K is a number field, and $\overline{\omega}^2_{C/K,a}$ in the function field case. We will recall the definitions of the stable Faltings height and the self-intersection of the admissible dualizing sheaf later on. For now, they can simply be thought of as meaningful arithmetic invariants attached to C.

Let us give some Diophantine motivations for this theorem. With K a number field, a basic remark is that the self-intersection number $\overline{\omega}^2_{C/K,a}$ is intimately related to the (classical) Bogomolov conjecture; one knows that it is nonnegative and that its positivity for a given curve C is equivalent to the Bogomolov conjecture for C ([Fal84], [Szp90, Theorem 3], [Zha93, Theorem 5.5]). The Bogomolov conjecture was proven by Ullmo [Ull98] using different methods, so one may raise the question about whether $\overline{\omega}^2_{C/K,a}$ is uniformly positive. This theorem answers that question affirmatively. Moreover, upper and lower bounds for $\overline{\omega}^2_{C/K,a}$ are related to bounds on the heights of "large" and "small" points of $C(K)$, respectively. For example, a good enough upper bound on $\overline{\omega}^2_{\mathcal{C}/\mathcal{O}_K,Ar}$ would[3] imply an *effective* version of the Mordell conjecture by work of Moret-Bailly [MB90, Théorème 5.2], which in turn implies Szpiro's discriminant conjecture and the abc conjecture [MB90, Sections 7, 8].

To briefly sketch the type of upper bound in question, we have Moret-Bailly's arithmetic Noether formula

$$12h_{Fal}(C) = \omega^2_{\mathcal{C}/\mathcal{O}_K,Ar} + \sum_{v \in M_K} \delta_v \log(n_v) - 4g[K : \mathbf{Q}] \log(2\pi)$$

where δ_v is the number of singularities in the fiber C_v (resp. the Faltings δ invariant of the base change C_v) if v is a finite place (resp. infinite place). Also, n_v is $\#k(v)$ (resp. $e \approx 2.718$, resp. e^2) if v is a finite place of K and $k(v)$ is the size of the residue field (resp. real, resp. complex). Then a bound

$$\omega^2_{\mathcal{C}/\mathcal{O}_K,Ar} \leq (12 - \epsilon) h_{Fal}(C) \tag{1.1.1}$$

for some $\epsilon > 0$ would imply an arithmetic Bogomolov–Miyaoka–Yau inequality [MB90, Hypothèse BM]

$$\omega^2_{\mathcal{C}/\mathcal{O}_K,Ar} \leq \left(\frac{12 - \epsilon}{\epsilon}\right) \left(\sum_{v \in M_K} \delta_v \log(n_v) - 4g[K : \mathbf{Q}] \log(2\pi)\right)$$

[3]Note that this is the *Arakelov* dualizing sheaf of a semistable model $\mathcal{C}$ of C spread out over $\mathcal{O}_K$, not the *admissible* dualizing sheaf of C over K; in particular the metrics at finite places are different. But the self-intersections of the two are related by a formula of Zhang [Zha93, Theorem 5.5].

which implies all the aforementioned conjectures. Such an inequality is still conjectural, and in fact [BMMB90] shows that the "naive" version of such an inequality (see for instance [Par88, Equation (9)]), constructed directly from the known geometric Bogomolov–Miyaoka–Yau inequality, cannot hold. We also note that the above bound (1.1.1) is just for illustrative purposes and is not at all what one would actually expect; the right-hand side of (1.1.1) should definitely involve more constants such as g, $[K : \mathbf{Q}]$, and $\log|\mathrm{disc}(K)|$.[4] For more background we will refer to the nice paper of Parshin [Par88] where the precise relations between the known (geometric) Bogomolov–Miyaoka–Yau inequality and the conjectured arithmetic one are explained in detail.

We now introduce the notations for the second theorem we want to discuss. For C a smooth curve over a number field K, and a fixed divisor $e \in \mathrm{Div}(C)$ of degree 1 defined over K, we can define two 1-dimensional cycles based on this data:

Definition 1.1.2.

(1) The *Gross-Schoen cycle* $GS_e(C) \in \mathrm{CH}_1(C^3)$ is defined to be the modified diagonal cycle in C^3. This means that for I a subset of $\{1, 2, 3\}$, we let Δ_I be the image of the embedding $C \hookrightarrow C^3$ given by $x \mapsto (a_1, a_2, a_3)$ where $a_i = x$ if $i \in I$ and $a_i = e$ otherwise. So for instance, Δ_{123} is given by the embedding $x \mapsto (x, x, x)$ and Δ_{12} is given by the embedding $x \mapsto (x, x, e)$. Then $GS_e(C)$ is defined to be the cycle

$$GS_e(C) \coloneqq \Delta_{123} - \Delta_{12} - \Delta_{13} - \Delta_{23} + \Delta_1 + \Delta_2 + \Delta_3.$$

(2) The *Ceresa cycle* $Ce_e(C) \in \mathrm{CH}_1(J(C))$ is given by $\iota_e(C) - [-1]_*\iota_e(C)$ where $\iota_e(C)$ is the Abel-Jacobi embedding $C \to J(C)$, $x \mapsto x - e$.

Note that the definitions of these cycles does depend on the choice of e. We also fix a choice of divisor e_{can} satisfying $(2g - 2)e_{can} \sim \omega_C$, which can be taken to be defined over the base field K at the cost of enlarging it. In this case we set $GS(C) \coloneqq GS_{e_{can}}(C)$ and similarly for $Ce(C)$.

These cycles are important for the following reason: for a smooth projective variety X over $\mathbf{C}$ of dimension d, and for each $0 \leq r \leq d$ there is a *cycle class map* $CH^r(X) \to H^{2r}(X, \mathbf{C})$ where H^* is some Weil cohomology theory, for instance the singular or de Rham cohomology. Then it is a fact that the Gross–Schoen cycle and Ceresa cycle are both *homologically trivial*, meaning that they lie in the kernel of the cycle class map (for the appropriate r). On the other hand, these cycles might not be *algebraically trivial*, meaning that they

[4] It is reasonable to allow dependency on $\log|\mathrm{disc}(K)|$, at least if we work in analogy to the geometric setting where there is a $(2g(B) - 2)\log(q)$ term if C extends to a fibered surface over a curve $B/\mathbf{F}_q$, because the latter is essentially -2 times the Euler characteristic of $\mathcal{O}_B$, and the same is true for the former for the Arakelov Euler characteristic. This is explained in [Par88, Section 2].

might not be 0 in the rational Chow group. For instance, that the Ceresa cycle of a "general" curve of genus at least 3 is algebraically nontrivial is due to Ceresa (1983). Therefore it is interesting to ask for when these cycles are actually algebraically trivial.

To do this, a conjectural height pairing $\langle \cdot, \cdot \rangle_{BB}$ on cycles was proposed by Beilinson and Bloch to study homologically trivial cycles. It is defined unconditionally in certain cases, such as in the case of the Gross–Schoen cycle by [Zha10]. The pairing has the property that $\langle D, D \rangle_{BB}$ for a cycle D vanishes if and only if D is algebraically trivial. We may introduce some notation:

Definition 1.1.3. Let $g \geq 2$ and let $\mathcal{M}_g$ denote a fine moduli space of curves of genus g over $\mathbf{Q}$ with universal family $\mathcal{C}_g$ over it.[5] We define the function $h_{GS} : \mathcal{M}_g(\overline{\mathbf{Q}}) \to \mathbf{R}$ by $[C] \mapsto \langle GS(C), GS(C) \rangle_{BB}$. We likewise define the function $h_{Ce} : \mathcal{M}_g(\overline{\mathbf{Q}}) \to \mathbf{R}$ by $[C] \mapsto \langle Ce(C), Ce(C) \rangle_{BB}$.

It is a fact that $h_{GS} = 6h_{Ce}$, so studying one of the heights is sufficient. We may now state the second main result, due to Gao–Zhang:

Theorem 1.1.4. [GZ24, Theorem 1.1] For each $g \geq 3$, there exists a Zariski open and dense subscheme U of $\mathcal{M}_g$ and positive numbers $\epsilon, c > 0$ (depending only on g) such that for any $s \in U(\overline{\mathbf{Q}})$ and any $e \in \mathrm{Pic}^1(\mathcal{C}_s)$, we have

$$\langle GS_e(\mathcal{C}_s), GS_e(\mathcal{C}_s) \rangle_{BB} \geq \epsilon(h_{Fal}(\mathcal{C}_s) + h_{NT}(e - e_{can,s})) - c$$

and

$$\langle Ce_e(\mathcal{C}_s), Ce_e(\mathcal{C}_s) \rangle_{BB} \geq \epsilon(h_{Fal}(\mathcal{C}_s) + h_{NT}(e - e_{can,s})) - c.$$

Here, h_{NT} is the Néron-Tate height on $\mathrm{Jac}(\mathcal{C}_s)(\overline{\mathbf{Q}})$. In particular when $e = e_{can,s}$, we have the special case

$$h_{GS}(s) \geq \epsilon h_{Fal}(\mathcal{C}_s) - c$$

and

$$h_{Ce}(s) \geq \epsilon h_{Fal}(\mathcal{C}_s) - c.$$

In particular, due to the same properties of the Faltings and Néron-Tate heights, the above result shows that h_{GS} and h_{Ce} have a lower bound on U and also satisfy a suitable Northcott property (but on algebraic points of $\mathrm{Pic}^1(\mathcal{C}_g/\mathcal{M}_g)$, so I don't want to set up this notation as it is not necessary for the discussion).

Remark 1.1.5. [GZ24] also shows that the open subscheme U of $\mathcal{M}_g$ is in fact defined over $\mathbf{Q}$, but we will not explain this part in these notes.

[5] Here $\mathcal{M}_g$ should be taken to be a Deligne–Mumford stack so that it is possible to define a universal family $\mathcal{C}_g$ over it. On the other hand, I am not very comfortable with the language of stacks, so I will usually just "pretend" that it is a scheme. In any case one can bypass this issue by adding in a suitable level structure to get a fine moduli space, which we will do below.

Remark 1.1.6. The main result of [Zha10] relates $\langle GS(C), GS(C)\rangle_{BB}$ with $\overline{\omega}^2_{C/K,a}$ in a way such that finding an upper bound for $\langle GS(C), GS(C)\rangle_{BB}$ in the form of an arithmetic Bogomolov–Miyaoka–Yau inequality would also imply the effective Mordell conjecture. This is a more Diophantine reason to study the Beilinson–Bloch height of the Gross–Schoen cycle.

1.2 A brief outline

Here we give a very basic sketch of the proofs in order to highlight the common theme that will be presented in this talk. In both cases the relevant quantities that need bounding are interpreted as *heights associated to a suitable adelically metrized line bundle.* The crux of the proof is determining the *bigness* of some of these line bundles, where bigness of an adelic line bundle $\overline{L}$ on a quasiprojective variety X (either over $\mathbf{Z}$, or over a function field of one variable) of absolute dimension d is defined as the condition that

$$\widehat{\mathrm{vol}}(X, \overline{L}) \coloneqq \lim_{m\to\infty} \frac{d!}{m^d} \widehat{h}^0(X, m\overline{L})$$

is strictly positive (of course, we haven't defined what the arithmetic analogue of the dimension of global sections $\widehat{h}^0$ is, and moreover it must be proved that this limit actually exists). We will prove the bigness of the following fundamental adelic line bundles on $\mathcal{M}_g$: the Deligne pairing $\pi_*\langle \omega_{\mathcal{C}_g/\mathcal{M}_g}, \omega_{\mathcal{C}_g/\mathcal{M}_g}\rangle$, whose height gives the $\overline{\omega}^2_{C/K,a}$ fiber-by-fiber; a suitable twist $\overline{\lambda}_{\mathcal{M}_g} + \mathcal{O}(c)$ of the Hodge bundle, whose height gives the Faltings height plus some constant $c > 0$; and a certain line bundle $\frac{2g+1}{2g-2}\pi_*\langle \omega_{\mathcal{C}_g/\mathcal{M}_g}, \omega_{\mathcal{C}_g/\mathcal{M}_g}\rangle + \mathcal{O}(\Phi_{\mathcal{M}_g})$, whose height gives the "Gross-Schoen height" h_{GS} fiber-by-fiber.

Now, following the philosophy of Arakelov geometry, the height function $h_{\overline{L}}$ corresponding to $\overline{L}$ is given by an (arithmetic) intersection number.[6] The key input is then the *height inequality* of Yuan–Zhang [YZ24, Theorem 5.3.7], which relies on the framework of adelic line bundles:

Theorem 1.2.1. Let $\pi : X \to S$ be a morphism of quasiprojective varieties over K, which is either a number field (in which case we set $k = \mathbf{Z}$) or a function field of one variable over another field k. Let $\overline{L} \in \widehat{\mathrm{Pic}}(X/k)$ and $\overline{M} \in \widehat{\mathrm{Pic}}(S/k)$ be adelic line bundles, and write $\widetilde{L}$ for the image of $\overline{L}$ in $\widehat{\mathrm{Pic}}(X/K)$.[7]

(1) If $\overline{L}$ is big on X/k, then there exist $\epsilon > 0$ and a nonempty open subvariety U of X such that $h_{\overline{L}}(x) \geq \epsilon h_{\overline{M}}(\pi(x))$ for all $x \in U(\overline{K})$ (notice that there are no assumptions on $\overline{M}$ here!).

(2) If $\overline{L}$ is nef on X/k, and $\widetilde{L}$ is big on X/K, then for any $c > 0$ there exist $\epsilon > 0$ and a nonempty open subvariety U of X such that $h_{\overline{L}}(x) \geq \epsilon h_{\overline{M}}(\pi(x)) - c$ for all $x \in U(\overline{K})$.

[6] For a quick reference for the classical case, see for instance [Mor14, Section 9.1], or Section 2.2.5 below.
[7] The notation can get a bit tricky, so it's important to be careful.

(3) If $\widetilde{L}$ is big on X/K, then there exist $c, \epsilon > 0$ and a nonempty open subvariety U of X such that $h_{\overline{L}}(x) \geq \epsilon h_{\overline{M}}(\pi(x)) - c$ for all $x \in U(\overline{K})$.

Applying this theorem to the aforementioned line bundles will give Theorems 1.1.1 and 1.1.4 (namely, parts (1) and (3) respectively).

Remark 1.2.2. The same type of method (proving the bigness of some relevant line bundle and applying the height inequality) can be used to prove a uniform version of the Bogomolov conjecture, which is the main result as mentioned in the title of [Yua24].

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2 Background on Arakelov theory and adelic line bundles

To discuss the above results we need the framework of adelic line bundles on quasiprojective varieties (e.g. $\mathcal{M}_g$) as developed by [YZ24], which is based on the projective case due to Zhang ([Zha93], [Zha95]). We will do this by first reviewing Arakelov theory. Due to lack of time and space, proofs in this section will be entirely omitted.

2.1 Height machine

We begin by briefly reviewing Weil's height machine.

Definition 2.1.1 (Naive height on projective space, number field case)**.** The naive height function $h : \mathbf{P}^n(\overline{\mathbf{Q}}) \to \mathbf{R}$ is defined by

$$h(x_0, \ldots, x_n) = \frac{1}{[K : \mathbf{Q}]} \sum_{v \in M_K} e_v \log \max\{\|x_0\|_v, \ldots, \|x_n\|_v\}$$

where K is a number field containing the coordinates $x_0, \ldots, x_n$, and $e_v = 2$ if v is a complex place and is 1 otherwise. The non-Archimedean absolute values are normalized by $\|x_v\| = n_v^{-\mathrm{ord}_v(x)}$ where $n_v \coloneqq \|\mathcal{O}_K/\mathfrak{p}_v\|$.

By the product formula this does not depend on the choice of coordinates x_i, and so $h(x_0, \ldots, x_n)$ is nonnegative because we can always scale so that one of the coordinates is 1. Moreover, h is Galois-invariant and enjoys the *Northcott property*: for any $B \geq 0, D \geq 1$, $\{x \in \mathbf{P}^n(\mathbf{Q}) : h(x) \leq B, [\mathbf{Q}(x) : \mathbf{Q}] \leq D\}$ is finite.

Theorem 2.1.2 (Weil's height machine)**.** Now let X be a projective variety defined over $\overline{\mathbf{Q}}$, and let $\mathbf{R}^{X(\overline{\mathbf{Q}})}$ be the set of real-valued functions on $X(\overline{\mathbf{Q}})$. Let $O(1)$ be the subset of bounded functions in $\mathbf{R}^{X(\overline{\mathbf{Q}})}$, and we agree to call two functions in $\mathbf{R}^{X(\overline{\mathbf{Q}})}$ *equivalent* if they are in the same class in $\mathbf{R}^{X(\overline{\mathbf{Q}})}/O(1)$. Then there exists a unique group homomorphism

$$\mathbf{h} : \mathrm{Pic}(X) \to \mathbf{R}^{X(\overline{\mathbf{Q}})}/O(1)$$

such that for any ample line bundle L on X and any map $i : X \to \mathbf{P}^N_{\overline{\mathbf{Q}}}$ with $i^*\mathcal{O}(1) \cong mL$ for some positive integer m, the function $h \circ i$ is equivalent to $m\mathbf{h}_L$.

The construction of $\mathbf{h}$ more or less proceeds from the desired conditions: for any very ample line bundle L we may choose a generating set of global sections that give a closed immersion $i : X \hookrightarrow \mathbf{P}^N$, and simply set $\mathbf{h}_L := h \circ i$. The actual function $h \circ i$ depends on the choice of these global sections, but its equivalence class does not (see [BG06, Chapter 2.2] for details). For ample line bundles we use the fact that mL is very ample for some positive integer m, so that we define $\mathbf{h}_L$ to be $\mathbf{h}_{mL}/m$. Finally, we use [Har77, Ex II.7.5(b)] to write any line bundle in $\mathrm{Pic}(X)$ as a difference of two ample line bundles and extend $\mathbf{h}$ by linearity. Of course we are missing many details, most glaringly that all of the constructions are independent of choices (up to $O(1)$).

Proposition 2.1.3. The height machine has the following properties:

(1) (Functoriality) If $f : X \to Y$ is a morphism of projective varieties over $\overline{\mathbf{Q}}$, and L is a line bundle on Y, then $\mathbf{h}_L \circ f \equiv \mathbf{h}_{f^*L}$. Here "$\equiv$" means "up to a bounded function".

(2) (Lower bound) If s is a global section of L, then $\mathbf{h}_L$ is bounded below on $(X - |\mathrm{div}(s)|)(\overline{\mathbf{Q}})$.

(3) (Nortchott property) If L is ample, and X is defined over a number field K, then for any $B \geq 0, D \geq 1$, the set $\{x \in X(K') : [K' : K] \leq D, \mathbf{h}_L(x) \leq B\}$ is finite.

The proofs of the above can be found in many textbooks, for instance [BG06, Chapter 2] or [HS00, Part B].

Since we will eventually also work in the function field case, we need to explain the naive height in this case as well. Let K be a function field of one variable over a field k. Then

Definition 2.1.4 (Naive height on projective space, function field case)**.** The naive height function $h : \mathbf{P}^n(\overline{K}) \to \mathbf{R}$ is defined by

$$h(x_0, \ldots, x_n) = \frac{1}{[K' : K]} \sum_{v \in M_{K'}} \log \max\{\|x_0\|_v, \ldots, \|x_n\|_v\}$$

where K is a number field containing the coordinates $x_0, \ldots, x_n$. The absolute values are normalized by $\|x_v\| = \exp(-[k(v) : k]\mathrm{ord}_v(x))$ where $k(v)$ is the residue field at v.

2.2 Heights via classical Arakelov theory

One of the key ideas of Arakelov theory is that heights should be geometrically interpreted as intersection numbers. Below we sketch the main theorems that we will need. This is mostly taken from Guo–Yuan's *A First Course in Arakelov Geometry* book [GY25].

Definition 2.2.1. An *arithmetic variety* $\mathcal{X}$ is a projective and flat integral scheme over $\mathrm{Spec}(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of a number field K. For notation, write $K(\mathbf{C})$ for the set of field embeddings $K \hookrightarrow \mathbf{C}$.

In the classical Arakelov theory, an intersection theory is defined on Hermitian line bundles. There is a more general theory of intersections of arithmetic Chow cycles, but we will not need it.

2.2.1 Preliminaries

Definition 2.2.2. Let X be a (projective) complex analytic space. A smooth/continuous *Hermitian line bundle* on X is a pair $\overline{L} \coloneqq (L, \|\cdot\|)$ where L is a line bundle on X and $\|\cdot\|$ is an assignment of a smooth/continuous Hermitian metric $\|\cdot\|_x$ to each fiber $L(x)$ as a one-dimensional vector space over $k(x)$. Here, "smooth/continuous" means that the metric varies smoothly/continuously in x; i.e. for all locally defined sections s of $\mathcal{O}_X$, $\|s(x)\|^2$ is smooth as a function of x. If we want to emphasize the "inner product" aspect of the Hermitian metric, we may write it as $h(\cdot, \cdot)$ instead.

For simplicity, our convention in the rest of this Section 2.2 is that metrics will be *smooth*, unless otherwise stated. Note that this agrees with the conventions of [Mor14] and [GY25], but differs from the conventions of [YZ24] where the metrics only need to be continuous.

We say that the *first Chern form*, or *curvature*, of $\overline{L}$ is

$$c_1(\overline{L}) = dd^c(-\log \|s\|) = \frac{i}{2\pi}\partial\overline{\partial}(-\log(h(s,s)))$$

defined locally for any nonvanishing local section s of $\mathcal{O}_X$.[8] This glues to a $(1,1)$-form on X, which is invariant under complex conjugation, hence real.

In the arithmetic setting, this becomes:

[8] Note that there are different normalizations floating around in the literature, which might differ by a factor of -1 or 2. Here we take the normalization $d^c = (\partial - \overline{\partial})/(2\pi i)$, so $dd^c = i\partial\overline{\partial}/\pi$.

Definition 2.2.3. Let $\mathcal{X}$ be an arithmetic variety. A *Hermitian line bundle* on $\mathcal{X}$ is a pair $\overline{\mathcal{L}} \coloneqq (\mathcal{L}, (|\cdot|_\sigma)_{\sigma \in K(\mathbf{C})})$ where $\mathcal{L}$ is a line bundle on $\mathcal{X}$ and $(\mathcal{L}_\sigma, \|\cdot\|_\sigma)$ is a (smooth) Hermitian line bundle over the complex algebraic variety $\mathcal{X}_\sigma \coloneqq \mathcal{X} \times_{\operatorname{Spec}(\mathcal{O}_K)} \operatorname{Spec}(\mathbf{C})$.[9] Here $\operatorname{Spec}(\mathbf{C})$ is a $\operatorname{Spec}(\mathcal{O}_K)$-scheme via $\sigma : K \hookrightarrow \mathbf{C}$. We also require compatibility of the metrics under complex conjugation, i.e. the complex conjugation map $X_\sigma \to X_{\overline{\sigma}}$ should be compatible with the metrics $\|\cdot\|_\sigma$, $\|\cdot\|_{\overline{\sigma}}$.[10]

There are obvious notions of isometry, dual, and tensor of Hermitian line bundles. There is also an obvious trivial Hermitian line bundle with underlying line bundle $\mathcal{O}_\mathcal{X}$ in the arithmetic case. This data gives us the *arithmetic Picard group* $\widehat{\operatorname{Pic}}(\mathcal{X})$, whose elements are isometry classes of Hermitian line bundles on $\mathcal{X}$. There is also an evident way to pull back Hermitian line bundles.

For future reference we will define the set of *effective/small sections* of a Hermitian line bundle.

Definition 2.2.4. For a Hermitian line bundle $\overline{\mathcal{L}}$ over $\mathcal{X}$, set

$$\widehat{H}^0(\mathcal{X}, \overline{\mathcal{L}}) \coloneqq \{s \in \Gamma(\mathcal{X}, \mathcal{L}) : \|s(x)\|_\sigma \leq 1 \text{ for all } x \in \mathcal{X}_\sigma,\, \sigma \in K(\mathbf{C}).\}$$

Also, define

$$\widehat{h}^0(\mathcal{X}, \overline{\mathcal{L}}) \coloneqq \log \#\widehat{H}^0(\mathcal{X}, \overline{\mathcal{L}}).$$

Note that $\#\widehat{H}^0(\mathcal{X}, \overline{\mathcal{L}}) < \infty$ because this is the set of elements with norm bounded by 1 in a $\mathbf{Z}$-lattice.

We think of $\widehat{H}^0(\mathcal{X}, \overline{\mathcal{L}})$ as the correct analogue of "global sections" of the Hermitian line bundle $\mathcal{L}$. They will be needed later for various positivity properties of our line bundles. Also, because we are no longer working over a field, the logarithm takes the place of the "dimension" function (compare the situation over a finite field, where for a finite-dimensional vector space V over a finite field $\mathbf{F}$, $\log \#V$ is exactly $\dim_\mathbf{F} V$ up to a constant).

We now turn to the language in terms of divisors. For us all divisors will be Cartier divisors unless stated otherwise.

Definition 2.2.5. Let X be a (projective) complex analytic space and D a divisor on X. We say $g : X - \operatorname{supp}(D) \to \mathbf{R}$ is a (smooth/continuous) *Green's function* for D if for any meromorphic function on an open subset $U \subseteq X$ such that $\operatorname{div}(f) = D|_U$, $g + \log|f|$ can be extended to a (smooth/continuous) function on U. We call the pair $\overline{D} = (D, g)$ a *Green divisor*. The *first Chern form* of $\overline{D}$ is locally defined to be

$$c_1(\overline{D}) = dd^c(g + \log|f|) = dd^c(g).$$

It does not depend on the choices of U and f.

[9] Abusing notation, as we really mean the complex analytic space associated to $X_\sigma(\mathbf{C})$.

[10] In other literature this is called a "real type" condition.

In the arithmetic setting:

Definition 2.2.6. Let $\mathcal{X}$ be an arithmetic variety. An *arithmetic divisor* on $\mathcal{X}$ is a pair $\overline{\mathcal{D}} = (\mathcal{D}, (g_\sigma)_{\sigma \in K(\mathbf{C})})$ where $\mathcal{D}$ is a divisor on X and g_σ is a *smooth* Green's function for the divisor D_σ of X_σ. We also require that g_σ is invariant under complex conjugation on X_σ, i.e. the map $X_\sigma \to X_{\overline{\sigma}}$ is compatible with the functions g_σ, $g_{\overline{\sigma}}$ in the evident way.

Definition 2.2.7. We say that an arithmetic divisor $(\mathcal{D}, (g_\sigma)_{\sigma \in K(\mathbf{C})})$ is *effective* if $\mathcal{D}$ is effective on $\mathcal{X}$ and each g_σ is nonnegative wherever it is defined on X_σ. If moreover each g_σ is strictly positive we say the arithmetic divisor is *strictly effective*. We write $\overline{\mathcal{D}}_1 \geq \overline{\mathcal{D}}_2$ if $\overline{\mathcal{D}}_1 - \overline{\mathcal{D}}_2$ is effective.

Definition 2.2.8. For $f \in K(\mathcal{X})^\times$, we define $\widehat{\operatorname{div}}(f)$ to be the arithmetic divisor

$$(\operatorname{div}(f), (-\log|f \otimes_\sigma 1|)_\sigma).$$

Here $f \otimes_\sigma 1$ is a meromorphic function on $\mathcal{X}_\sigma$ and $|\cdot|$ is the usual absolute value on $\mathbf{C}$. We call any arithmetic divisor of the form $\widehat{\operatorname{div}}(f)$ a *prinicipal arithmetic divisor*.

As before there is an obvious abelian group of arithmetic divisors $\widehat{\operatorname{Div}}(\mathcal{X})$ with identity element $(0, 1)$. It has a subgroup $\widehat{\operatorname{Pr}}(\mathcal{X})$ consisting of the principal arithmetic divisors, and we set $\widehat{\operatorname{Cl}}(\mathcal{X})$ to be the quotient, called the *arithmetic divisor class group*.

As in the classical case there is a comparison between arithmetic divisors and Hermitian line bundles. For $\overline{\mathcal{D}} = (\mathcal{D}, (g_\sigma))$ we define

$$\mathcal{O}(\overline{\mathcal{D}}) := (\mathcal{O}(\mathcal{D}), (\|\cdot\|_\sigma))$$

with $\|s_{\mathcal{D}} \otimes_\sigma 1\|_\sigma = \exp(-g_\sigma)$, where $s_{\mathcal{D}}$ is the canonical section of $\mathcal{O}(\mathcal{D})$ corresponding to $1 \in K(\mathcal{X})^\times$. Conversely for a Hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|_\sigma)$, we may define an arithmetic divisor for any nonzero meromorphic section s of $\mathcal{L}$:

$$\widehat{\operatorname{div}}(s) := (\operatorname{div}(s), (-\log\|s\|_\sigma)).$$

It is clear that the class of this divisor in $\widehat{\operatorname{Cl}}(\mathcal{X})$ does not depend on the choice of s, so we get a map $\widehat{\operatorname{Cl}}(\mathcal{X}) \to \widehat{\operatorname{Pic}}(\mathcal{X})$. Then we have:

Proposition 2.2.9. The above map $\widehat{\operatorname{Cl}}(\mathcal{X}) \to \widehat{\operatorname{Pic}}(\mathcal{X})$ is an isomorphism of groups.

By abuse of notation, for $\overline{\mathcal{D}}$ an arithmetic divisor, we write $\widehat{H}^0(X, \overline{\mathcal{D}})$ for $\widehat{H}^0(X, \mathcal{O}(\overline{\mathcal{D}}))$, and similarly for $\widehat{h}^0$. It is easy to see that

$$\widehat{H}^0(X, \overline{\mathcal{D}}) = \{0\} \cup \{f \in K(\mathcal{X})^\times : \widehat{\operatorname{div}}(f) + \overline{\mathcal{D}} \text{ is effective}\}.$$

2.2.2 (Top) Intersection theory

We can now do the intersection theory in the case of line bundles. A reference for this section is [YZ24, Appendix A.3].

Suppose $\mathcal{X}$ is an arithmetic variety of total dimension d (not relative dimension over $\mathcal{O}_K$). We would like to define a top intersection number

$$\overline{\mathcal{L}}_1 \cdot \overline{\mathcal{L}}_2 \cdots \overline{\mathcal{L}}_d \coloneqq \widehat{\deg}(\overline{\mathcal{L}}_1 \cdot \overline{\mathcal{L}}_2 \cdots \overline{\mathcal{L}}_d) : \widehat{\mathrm{Pic}}(\mathcal{X})^d \to \mathbf{R}.$$

We do this by induction. First, when $d = 1$, $\mathcal{X}$ is an arithmetic curve, i.e. $\mathrm{Spec}(R)$ for R an order in K. Then $\deg(\mathcal{L}_1)$ is just the usual Arakelov degree of an arithmetic divisor $\mathcal{D} + \sum_{\sigma \in K(\mathbf{C})} n_\sigma[\sigma]$ corresponding to $\mathcal{L}_1$,

$$\widehat{\deg}\left(\mathcal{D} + \sum_{\sigma \in K(\mathbf{C})} n_\sigma[\sigma]\right) = \sum_{\mathfrak{p} \text{ closed point of } \mathcal{X}} \mathrm{ord}_{\mathfrak{p}}(\mathcal{D}) \log(n_{\mathfrak{p}}) + \sum_{\sigma \in K(\mathbf{C})} n_\sigma.$$

Now suppose we have made the definition in dimension $d-1$, $d > 1$. For s_d a nonzero meromorphic section of $\mathcal{L}_d$, so that $\mathrm{div}(s_d)$ as a Weil divisor is $\sum_i a_i \mathcal{Z}_i$ and $\mathcal{O}(\widehat{\mathrm{div}}(s_d)) \cong \overline{\mathcal{L}}_d$, we define

$$\overline{\mathcal{L}}_1 \cdot \overline{\mathcal{L}}_2 \cdots \overline{\mathcal{L}}_d \coloneqq \sum_i a_i \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{d-1} \cdot \mathcal{Z}_i - \sum_{\sigma \in K(\mathbf{C})} \int_{X_\sigma} \log \|s_d\|_\sigma \, c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_{d-1}). \tag{2.2.1}$$

We briefly explain the terms. Each prime Weil divisor $\mathcal{Z}_i$ is either vertical or horizontal, meaning its image under the structure map to $\mathrm{Spec}(\mathcal{O}_K)$ is either a point or the whole scheme. In the vertical case, $\mathcal{Z}_i$ is a projective variety over a finite field $\mathbf{F}_q$, and $\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{d-1} \cdot \mathcal{Z}_i$ is just $(\mathcal{L}_1|_{\mathcal{Z}_i} \cdots \mathcal{L}_{d-1}|_{\mathcal{Z}_i}) \cdot \log(q)$ in terms of the geometric top intersection number of line bundles. In the horizontal case, $\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{d-1} \cdot \mathcal{Z}_i$ is just $\mathcal{L}_1|_{\mathcal{Z}_i} \cdots \mathcal{L}_{d-1}|_{\mathcal{Z}_i}$, which is defined by induction. Finally, note that $c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_{d-1})$ is a $(d-1, d-1)$-dimensional form on the $d-1$-dimensional complex algebraic variety (thought of as a complex manifold), so the integral makes sense.

As in the classical case, we have the following theorem:

Theorem 2.2.10. In the above notation, our rule for the intersection number $\overline{\mathcal{L}}_1 \cdot \overline{\mathcal{L}}_2 \cdots \overline{\mathcal{L}}_d : \widehat{\mathrm{Pic}}(\mathcal{X})^d \to \mathbf{R}$ is well-defined: it is independent of the choice of the section s_d. Also, it is symmetric and multilinear.

If $\mathcal{Z}$ is an integral closed subscheme of $\mathcal{X}$ of absolute dimension e, then we can define $\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_e \cdot \mathcal{Z}$ as above, splitting into cases depending on whether $\mathcal{Z}$ is horizontal or vertical. Also, recall that a Hermitian line bundle $\overline{L}$ on a complex analytic space is said to be *positive* (resp. *semipositive*) if its first Chern form defines a positive (resp. semipositive) real

(1, 1)-form, meaning that in local coordinates with $c_1(\overline{L}) = i\sum_{i,j} h_{ij}(z)dz_i \wedge d\overline{z}_j$, the real Hermitian/symmetric matrix $(h_{ij}(z))$ is always positive definite (resp. positive semidefinite). We will need these for the fundamental notion of *nefness*.

Definition 2.2.11. Let $\mathcal{X}$ be an arithmetic variety and $\overline{\mathcal{L}}$ a Hermitian line bundle on $\mathcal{X}$. We say that $\overline{\mathcal{L}}$ is *nef* if:

(1) $\mathcal{L}$ is nef in the geometric sense, meaning that $\widehat{\deg}(\overline{\mathcal{L}} \cdot \mathcal{Z}) \geq 0$ for all 1-dimensional integral closed subschemes $\mathcal{Z} \subseteq \mathcal{X}$.

(2) Each $(\mathcal{L}_\sigma, \|\cdot\|_\sigma)$ is a semipositive Hermitian line bundle on X_σ.

There is a corresponding notion of nefness for arithmetic divisors, defined in the obvious way.

As in the classical case, nefness is a weaker positivity subsitute for ampleness, but it is also more versatile (there is a notion of ampleness for Hermitian line bundles, but we won't need it for the ultimate adelic line bundle theory).

Finally, there is also a projection formula as in the classical case:

Proposition 2.2.12 (Projection formula)**.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of arithmetic varieties over $\mathcal{O}_K$. Then if $\mathcal{Z}$ is an integral closed subscheme of $\mathcal{X}$ of absolute dimension e, and $\overline{\mathcal{L}}_1, \ldots, \overline{\mathcal{L}}_e$ are Hermitian line bundles on $\mathcal{Y}$, then

$$f^*\overline{\mathcal{L}}_1 \cdots f^*\overline{\mathcal{L}}_e \cdot \mathcal{Z} = \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_e \cdot f_*(\mathcal{Z}),$$

where $f_*(\mathcal{Z})$ is the pushforward of $\mathcal{Z}$ as a cycle. This makes sense because f is proper.

2.2.3 2-dimensional intersection theory

It might be helpful to give a more concrete construction when $\dim \mathcal{X} = 2$, i.e. the case of a relative curve. For simplicity we also assume that $\mathcal{X}$ is regular and the generic fiber $\mathcal{X}_K$ is geometrically integral. Write $\pi : \mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K)$ for the structure morphism.

We would like to explicitly give the intersection pairing $\widehat{\mathrm{Cl}}(\mathcal{X}) \times \widehat{\mathrm{Cl}}(\mathcal{X}) \to \mathbf{R}$ in terms of divisors. To start, assume two arithmetic divisors $\overline{\mathcal{D}}_1 = (\mathcal{D}_1, (g_{1,\sigma}))$ and $\overline{\mathcal{D}}_2 = (\mathcal{D}_2, (g_{2,\sigma}))$ on $\mathcal{X}$ intersect property, in the sense that the supports of $\mathcal{D}_1$ and $\mathcal{D}_2$ share no common component (here because $\mathcal{X}$ is regular we can take $\mathcal{D}_1$ and $\mathcal{D}_2$ to be Weil divisors, if that is more convenient). Then we define

$$\overline{\mathcal{D}}_1 \cdot \overline{\mathcal{D}}_2 = \mathcal{D}_1 \cdot \mathcal{D}_2 + \sum_{\sigma \in K(\mathbf{C})} \left(g_{1,\sigma}(\mathcal{D}_{2,\sigma}) + \int_{\mathcal{X}_\sigma} g_2 c_1(\overline{\mathcal{D}}_{1,\sigma}) \right). \tag{2.2.2}$$

To explain the terms, we reduce by linearity to the case where $\mathcal{D}_1$ and $\mathcal{D}_2$ are both effective. To explain the first term, we define $I_x(\mathcal{D}_1, \mathcal{D}_2)$ for a closed point $x \in \mathcal{X}$ to be the length of the $\mathcal{O}_{X,x}$-module $\mathcal{O}_{X,x}/(d_1, d_2)$, where $d_1, d_2 \in \mathcal{O}_{X,x}$ are local equations for $\mathcal{D}_1, \mathcal{D}_2$ at x. We consider the 0-cycle $I(D, E) \coloneqq \sum_{x \in \mathcal{X} \text{ closed}} I_x(\mathcal{D}_1, \mathcal{D}_2)[x]$, which is a finite sum because the coefficient of x is nonzero if and only if $x \in \operatorname{supp}(\mathcal{D}_1) \cap \operatorname{supp}(\mathcal{D}_2)$, which is 0-dimensional. Then we define

$$\mathcal{D}_1 \cdot \mathcal{D}_2 \coloneqq \deg(\pi_* I(D, E)) = \sum_{x \in \mathcal{X} \text{ closed}} I_x(\mathcal{D}_1, \mathcal{D}_2)[k(x) : k(\pi(x))] \log \# k(\pi(x)).$$

Next, $\mathcal{D}_{2,\mathbf{C}}$ is just a formal sum of points on the Riemann surface $\mathcal{X}_\sigma$, so $g_{1,\sigma}(\mathcal{D}_{2,\mathbf{C}})$ just means that we apply the Green's function $g_{1,\sigma}$ to points and extend via linearity. This makes sense because $g_{1,\sigma}$ is a Green's function for $\mathcal{D}_{1,\sigma}$, which is disjoint from $\mathcal{D}_{2,\sigma}$. The last term is self-explanatory.

Using the correspondence between arithmetic divisors and Hermitian line bundles on $\mathcal{X}$, it is easy to check that this explicit formula is exactly (2.2.1). Indeed, if $\overline{\mathcal{D}}_1$ and $\overline{\mathcal{D}}_2$ correspond to $\overline{\mathcal{L}}_1 = (\mathcal{O}(\mathcal{D}_1), \|\cdot\|_{1,\sigma}$ and $\overline{\mathcal{L}}_2 = (\mathcal{O}(\mathcal{D}_2), \|\cdot\|_{2,\sigma})$ respectively, then using the canonical section 1 of $\mathcal{O}(\mathcal{D}_2)$ with $\operatorname{div}(1) = \mathcal{D}_2 = \sum_i a_i \mathcal{Z}_i$, we have

$$\overline{\mathcal{L}}_1 \cdot \overline{\mathcal{L}}_2 = \sum_i a_i \overline{\mathcal{L}}_1 \cdot \mathcal{Z}_i - \sum_{\sigma \in K(\mathbf{C})} \int_{X_\sigma} -g_{2,\sigma} c_1(\overline{\mathcal{L}}_1)$$

If $\mathcal{Z}_i$ is vertical over $\mathfrak{p} \in \operatorname{Spec}(\mathcal{O}_K)$, then

$$\overline{\mathcal{L}}_1 \cdot \mathcal{Z}_i = \deg(\mathcal{L}_1|_{\mathcal{Z}_i}) \log(n_\mathfrak{p}) = \sum_{z \in \mathcal{Z}_i \text{ closed}} \left(\operatorname{ord}_{\mathcal{O}_{\mathcal{Z}_i,z}} \mathcal{D}_1\right) [k(z) : k(\mathfrak{p})] \log(n_\mathfrak{p}),$$

and if $\mathcal{Z}_i$ is horizontal,

$$\begin{aligned} \overline{\mathcal{L}}_1 \cdot \mathcal{Z}_i &= \widehat{\deg}(\mathcal{L}_1|_{\mathcal{Z}_i}) \\ &= \sum_{z \in \mathcal{Z}_i \text{ closed}} \operatorname{ord}_{\mathcal{O}_{\mathcal{Z}_i,z}} \mathcal{D}_1 \log(\# k(z)) + \sum_{\sigma \in K(\mathbf{C})} g_{1,\sigma}(\mathcal{Z}_{i,\sigma}) \\ &= \sum_{z \in \mathcal{Z}_i \text{ closed}} \operatorname{ord}_{\mathcal{O}_{\mathcal{Z}_i,z}} \mathcal{D}_1 [k(z) : k(\mathfrak{p})] \log(n_\mathfrak{p}) + \sum_{\sigma \in K(\mathbf{C})} g_{1,\sigma}(\mathcal{Z}_{i,\sigma}). \end{aligned}$$

Hence we recover Equation (2.2.2).

If we started with the definition in (2.2.2) it is not immediately clear that the term $g_{1,\sigma}(\mathcal{D}_{2,\sigma}) + \int_{\mathcal{X}_\sigma} g_2 c_1(\overline{\mathcal{D}}_{1,\sigma})$ is symmetric in $\overline{\mathcal{D}}_1, \overline{\mathcal{D}}_2$, but this can be proved using the Poincaré–Lelong formula. This is the approach taken in [Mor14, Section 4.2]. In fact, [Mor14, Chapter 4] provides a more explicit approach to the (very enjoyable and beautiful!) 2-dimensional Arakelov intersection theory along the lines of this subsection, so we will simply refer the reader there for more details. Chapter 5 of [GY25] also has these details.

2.2.4 Deligne pairing

The Deligne pairing is a *relative* version of the above intersection theory, where the output will be a line bundle instead of a number. Because it is rather complicated, especially in higher dimension, we can only sketch the construction and state the main results we need. We will make heavy use of the (properties of the) Deligne pairing later on, when we discuss adelic line bundles and the proofs of our main theorems. It was originally constructed in [Del87] for morphisms of relative dimension 1, and [Elk89] for morphisms of any relative dimension. Other references include [Mor14, Section 4.1] (in the 2-dimensional case only), [YZ24, Section 4.2], [GY25, Section 4.4], and [Dol22].

Remark 2.2.13. Suppose as a motivation that we have an extension of rings of integers of number fields, $\mathrm{Spec}(\mathcal{O}_L) \to \mathrm{Spec}(\mathcal{O}_K)$, which has relative dimension 0. Given a (vanilla) divisor D on $\mathrm{Spec}(\mathcal{O}_L)$, we can push the finite forward to $\mathrm{Spec}(\mathcal{O}_K)$ and take its degree, obtaining the "intersection number" of D on the "arithmetic curve $\mathrm{Spec}(\mathcal{O}_L)$ over $\mathrm{Spec}(\mathcal{O}_K)$." We ignore, for the sake of the motivation, that this does not make sense without any Archimedean contributions. Now if $\mathcal{L}$ is a line bundle on $\mathrm{Spec}(\mathcal{O}_L)$ corresponding to D, then to realize this number as the degree of a line bundle on $\mathrm{Spec}(\mathcal{O}_K)$ in a functorial way, we should take this line bundle to be the *norm* of $\mathcal{L}$, since pushing forward D is exactly taking the norm of the fractional ideal corresponding to D. This is the idea we want to generalize in the Deligne pairing, and it will turn out that the Deligne pairing in relative dimension 0 will just be the norm functor on line bundles, generalized appropriately to the Hermitian setting.

First, we consider a projective flat morphism $f : X \to Y$ between Noetherian integral schemes of pure relative dimension $n \geq 0$.[11] We first treat the underlying line bundles of the Deligne pairing. Denote by $\mathcal{P}\mathrm{ic}(X)$ and $\mathcal{P}\mathrm{ic}(Y)$ the groupoids of line bundles on X and Y.[12]

Theorem 2.2.14. There exists a canonical symmetric and multilinear functor

$$\mathcal{P}\mathrm{ic}(X)^{n+1} \to \mathcal{P}\mathrm{ic}(Y), \quad (L_1, \ldots, L_{n+1}) \mapsto f_*\langle L_1, \ldots, L_{n+1}\rangle$$

satisfying the following:

(1) For any morphism $Y' \to Y$ of Noetherian integral schemes, the pairing is compatible with the base change $f_{Y'} : X_{Y'} \to Y'$ in the evident way.

(2) If s_{n+1} is a global section of L_{n+1} which is not a zero-divisor at any point of X (i.e. that the sheaf map $\mathcal{O}_X \to L_{n+1}$ given by s_{n+1} is injective), and $Z \coloneqq \mathrm{div}(s_{n+1})$ is flat over Y, then there is a canonical isomorphism

$$f_*\langle L_1, \ldots, L_{n+1}\rangle \xrightarrow{\sim} (f|_Z)_*\langle L_1|_Z, \ldots, L_{n+1}|_Z\rangle$$

[11] It is definitely possible that some hypotheses can be relaxed, but we don't aim for full generality.

[12] I would prefer not to use the word "groupoid," but it is a convenient way of expressing the fact that the Deligne pairing is constructed via the individual line bundles first, and then it is shown that it respects isomorphism classes of line bundles.

(3) If $n = 0$, we have

$$f_*\langle L\rangle = N_{X/Y}(L),$$

the norm functor as in [Dol22, Definition A.7].

(4) If $s_1, \dots, s_{n+1}$ is a *strongly regular sequence* of global sections of the L_i (for the definition see [YZ24, Section 4.2.1]), then there is a canonical global section $\langle s_1, \dots, s_{n+1}\rangle$ of $f_*\langle L_1, \dots, L_{n+1}\rangle$.

As promised,

Theorem 2.2.15. Continuing the above notation, if X and Y are projective varieties over a field k, Y is a *curve*, and $L_1, \dots, L_{n+1}$ are line bundles on X, then

$$\deg(f_*\langle L_1, \dots, L_{n+1}\rangle) = \deg(L_1 \cdots L_{n+1}).$$

Remark 2.2.16. Sometimes the Deligne pairing will also be denoted $\langle L_1, \dots, L_{n+1}\rangle$ or $\langle L_1, \dots, L_{n+1}\rangle_{X/Y}$ if f is understood from context.

Remark 2.2.17. In the 2-dimensional case, the construction can be made extremely explicit. See [Mor14, Section 4.1].

We now turn to the Hermitian side of things. Let $f : X \to Y$ be a projective flat morphism of quasiprojective integral varieties over $\mathbf{C}$. We first consider $Y = \mathrm{Spec}(\mathbf{C})$, so that we want to equip a metric on the 1-dimensional vector space $f_*\langle \overline{L}_1, \dots \overline{L}_{n+1}\rangle$ where $\overline{L}_1, \dots, \overline{L}_1$ are continuous *integrable* Hermitian line bundles on X.[13] We do this inductively. By linearity we may assume that the L_i are very ample, so take a global section s_{n+1} of L_{n+1} that is not a zero-divisor. By item (2) of Theorem 2.2.14, we have a natural isomorphism of 1-dimensional vector spaces

$$[s_{n+1}] : f_*\langle L_1, \dots, L_{n+1}\rangle \xrightarrow{\sim} (f|_Z)_*\langle L_1|_Z, \dots, L_{n+1}|_Z\rangle.$$

We define the norm (as a linear map of normed vector spaces) of this isomorphism to be

$$\log \|[s_{n+1}]\| = -\int_X \log \|s_{n+1}\|\, c_1(\overline{L}_1) \cdots c_1(\overline{L}_n).$$

This determines the norm on $f_*\langle L_1, \dots, L_{n+1}\rangle$ by induction. It can be proved that this construction is indeed well-defined (i.e. independent of all choices made in the induction), and the resulting Deligne pairing of Hermitian line bundles is symmetric and multilinear. It

[13] If we are only working with continuous metrics, as [YZ24] does, instead of smooth metrics, we need to add some "integrability" condition here, meaning that the $\mathcal{L}_i$ can be written as differences of Hermitian line bundles with semipositive metrics. All smooth metrics are integrable, so we don't see this assumption in the smooth case.

is of course not an accident that this inductive formula for the metric is almost identical to the "infinite part" of the intersection product as defined in Section 2.2.2.

We now return to the setting of general Y. For a closed point $y \in Y$, there is a canonical metric $\|\cdot\|_{X_y}$ of $f_{y,*}\langle L_{1,y}, \ldots, L_{n+1,y}\rangle$ by the $Y = \mathrm{Spec}(\mathbf{C})$ case above. By functoriality this Deligne pairing is naturally isomorphic to the fiber $f_*\langle L_1, \ldots, L_{n+1}\rangle_y$, so we get a natural metric there. Varying y, we get a fiberwise-defined "metric" $\|\cdot\|_{X/Y}$ on $f_*\langle L_1, \ldots, L_{n+1}\rangle$, which we claim is continuous.

Theorem 2.2.18. Let $f : X \to Y$ be a projective flat morphism of pure relative dimension n of quasiprojective varieties over $\mathbf{C}$. Let $\overline{L}_1, \ldots, \overline{L}_{n+1}$ be Hermitian line bundles on X. Then

(1) The metric $\|\cdot\|_{X/Y}$ is a continuous metric on $f_*\langle L_1, \ldots, L_{n+1}\rangle$.

(2) The construction of $\|\cdot\|_{X/Y}$ is symmetric and multilinear in the $\overline{L}_i$.

(3) The metric $\|\cdot\|_{X/Y}$ is compatible with base changes $Y' \to Y$ of quasiprojective $\mathbf{C}$-varieties.

(4) If f is smooth, then $\|\cdot\|_{X/Y}$ is in fact smooth. In this case, we have an equality of $(1,1)$-forms

$$c_1\left(f_*\langle L_1, \ldots, L_{n+1}\rangle, \|\cdot\|_{X/Y}\right) = \int_{X/Y} c_1(\overline{L}_1) \cdots c_1(\overline{L}_{n+1}).$$

(5) If the metrics of the $\overline{L}_i$ are all semipositive, so is $\|\cdot\|_{X/Y}$.

Proof. See [YZ24, Section 4.3]. Note that their treatment is even more general, replacing "flat" in the hypothesis with "finite Tor-dimension." On the other hand, for most situations, "flat" is sufficient. □

We now put everything together in the arithmetic case. Let $\mathcal{X}$ and $\mathcal{Y}$ are arithmetic varieties over $\mathcal{O}_K$, and let $f : \mathcal{X} \to \mathcal{Y}$ be a flat morphism of relative dimension n. Given Hermitian line bundles $\overline{\mathcal{L}}_1, \ldots, \overline{\mathcal{L}}_{n+1}$ on $\mathcal{X}$, we can define a Hermitian line bundle with *a priori* continuous metric

$$f_*\langle \overline{\mathcal{L}}_1, \ldots, \overline{\mathcal{L}}_{n+1}\rangle := \left(f_*\langle \mathcal{L}_1, \ldots, \mathcal{L}_{n+1}\rangle, \left(\|\cdot\|_{\mathcal{X}_\sigma/\mathcal{Y}_\sigma}\right)\right).$$

If f is in addition smooth on the generic fiber, then by item (4) of Theorem 2.2.18 the Hermitian metric is actually smooth, and so we get a symmetric and $(n+1)$-linear arithmetic Deligne pairing $\widehat{\mathrm{Pic}}(\mathcal{X})^{n+1} \to \widehat{\mathrm{Pic}}(\mathcal{Y})$ (remember for this Section 2.2 we are assuming Hermitian metrics are smooth for simplicity). It has the following properties:

Theorem 2.2.19. Let $f : \mathcal{X} \to \mathcal{Y}$ be as above and assume for simplicity that f is smooth on the generic fiber. Then the Deligne pairing as constructed above has the following properties:

(1) If $\dim(\mathcal{Y}) = 1$, then
$$\widehat{\deg}(f_*\langle\overline{\mathcal{L}}_1, \ldots, \overline{\mathcal{L}}_{n+1}\rangle) = \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{d+1}.$$

(2) If the $\overline{\mathcal{L}}_i$ are all nef (see Definition 2.2.11), then so is $f_*\langle\overline{\mathcal{L}}_1, \ldots, \overline{\mathcal{L}}_{n+1}\rangle$.

Proof. Part (1) should be believable from the construction of the Deligne pairing. For the full proof see [YZ24, Lemma 4.4.1, Lemma 4.4.3]. □

As with the top intersection number, there are natural projection formulas that the Deligne pairing satisfies, which we will omit here.

2.2.5 Heights via intersection theory

In this part we give a brief overview of the Arakelov-theoretic interpretation of heights as intersection numbers.

Let $\mathcal{X}/\mathrm{Spec}(\mathcal{O}_K)$ be an arithmetic variety with generic fiber X/K. Let $\overline{\mathcal{L}}$ be a Hermitian line bundle on $\mathcal{X}$. We define the *height function associated to* $(\mathcal{X}, \mathcal{L})$ as follows:

Definition 2.2.20. The height function $h_{\overline{\mathcal{L}}} : X(\overline{K}) \to \mathbf{R}$ associated to $(\mathcal{X}, \mathcal{L})$ is given by
$$h_{\overline{\mathcal{L}}}(x) = \frac{\widehat{\deg}(\overline{\mathcal{L}} \cdot \overline{x})}{[K(x) : \mathbf{Q}]},$$
where $\overline{x}$ is the Zariski closure inside $\mathcal{X}$ of the point x in the generic fiber of $\mathcal{X}$.

Note that this definition is "absolute," in that the height does not change if we extend K. This agrees with the convention taken in [Mor14, Section 9.1].

In many applications we instead start with a projective variety X over a number field K instead an arithmetic variety $\mathcal{X}/\mathcal{O}_K$. This motivates the following definition, which will be very important later in the adelic line bundle theory:

Definition 2.2.21. Let X be a projective variety over a number field K, and L a line bundle on X. We say an *integral model* of X is an arithmetic variety $\mathcal{X}/\mathcal{O}_K$ along with an isomorphism $\mathcal{X}_K \to X$ of K-schemes. Note that this isomorphism is part of the data of an arithmetic model, but we frequently drop it from the notation. We also say an *arithmetic model* of (X, L) is a pair $(\mathcal{X}, \overline{\mathcal{L}})$ where $\mathcal{X}$ is an integral model of X, and $\overline{\mathcal{L}}$ is a Hermitian line bundle such that the pullback of $\mathcal{L}$ to X, *via the implicit isomorphism* $\mathcal{X}_K \to X$, is isomorphic to L.

We now relate these Arakelov heights to the Weil heights as defined in Theorem 2.1.2. To start, consider the following example:

Example 2.2.22. Let K be a number field, $X = \mathbf{P}^n_K$, $L = \mathcal{O}(1)_K$. We take $\mathcal{X} = \mathbf{P}^n_{\mathcal{O}_K}$ to be an integral model of X. For the arithmetic model of (X, L), we take $\overline{\mathcal{L}}$ to have underlying line bundle $\mathcal{O}(1)$ on $\mathcal{X}$, equipped with metric

$$\|s(x)\|_{can,\sigma} := \frac{|s(x_0, \ldots, x_n)|}{\max_i |x_i|}.$$

for $x = [x_0 : \ldots : x_n]$ a (classical) point in $\mathcal{X}_\sigma$ and s a section of $\mathcal{L}_\sigma$. We also call this Hermitian line bundle $\overline{\mathcal{O}(1)}_{can}$.

We claim that $h_{\overline{\mathcal{O}(1)}_{can}}$ is simply the naive height on $\mathbf{P}^n_K$. By base change, we may assume $x = [x_0 : \ldots : x_n]$ is a rational point in $\mathbf{P}^n_K(K)$. Let $\overline{x} : \operatorname{Spec}(\mathcal{O}_K) \to \mathcal{X}$ be the Zariski closure of x, and take a global section $s = a_0 t_0 + \ldots + a_n t_n$, $a_i \in \mathcal{O}_K$ of $\mathcal{O}(1)$ with no zero along $\overline{x}$. Then $\widehat{\operatorname{div}}(s) = (\operatorname{div}(s), (-\log \|s\|_\sigma))$, and

$$\widehat{\deg}(\overline{\mathcal{L}} \cdot \overline{x}) = \widehat{\deg}(\widehat{\operatorname{div}}(s|_{\overline{x}})) = \sum_{\mathfrak{p} \subseteq \mathcal{O}_K} \operatorname{ord}_{\mathfrak{p}}(s|_{\overline{x}}) \log(n_{\mathfrak{p}}) - \sum_{\sigma \in K(\mathbf{C})} \log \|s(x_\sigma)\|_{can,\sigma}.$$

For the first term $\operatorname{ord}_{\mathfrak{p}}(s|_{\overline{x}})$, we may work in $\mathcal{O}_{K,\mathfrak{p}}$. We claim it is equal to

$$\operatorname{ord}_{\mathfrak{p}}(s|_{\overline{x}}) = \operatorname{ord}_{\mathfrak{p}}(a_0 x_0 + \ldots + a_n x_n) - \min_i \operatorname{ord}_{\mathfrak{p}}(x_i). \tag{2.2.3}$$

Both quantities are invariant upon permuting the coordinates and scaling all the x_i by a nonzero element in $\mathcal{O}_{K,\mathfrak{p}}$, so we may assume upon dividing by an appropriate power of a uniformizer that $x_0 = 1$. In particular, $\operatorname{ord}_{\mathfrak{p}}(a_0 x_0 + \ldots + a_n x_n) - \min_i \operatorname{ord}_{\mathfrak{p}}(x_i) = \operatorname{ord}_{\mathfrak{p}}(a_0 + \ldots + a_n x_n)$. Then to compute $\operatorname{ord}_{\mathfrak{p}}(s|_{\overline{x}})$, we may do so under a trivialization $\mathcal{O}(1)|_U \cong \mathcal{O}_U$ where $U = D_+(t_0)$ is the standard affine chart, so that under this chart/trivialization, $\overline{x}$ has coordinates $(x_1, \ldots, x_n)$ and $s|_{\overline{x}}$ is given by $a_0 + a_1 x_1 + \ldots + a_n x_n \in \mathcal{O}_{K,\mathfrak{p}}$. It follows that

$$\operatorname{ord}_{\mathfrak{p}}(s|_{\overline{x}}) = \operatorname{ord}_{\mathfrak{p}}(a_0 + a_1 x_1 + \ldots + a_n x_n).$$

Therefore Equation 2.2.3 gives

$$\operatorname{ord}_{\mathfrak{p}}(s|_{\overline{x}}) \log(n_{\mathfrak{p}}) = (\operatorname{ord}_{\mathfrak{p}}(a_0 x_0 + \ldots + a_n x_n) - \min_i \operatorname{ord}_{\mathfrak{p}}(x_i) \log(n_{\mathfrak{p}}) = -\log|a_0 x_0 + \ldots + a_n x_n|_{\mathfrak{p}} + \log \max |x_i|_{\mathfrak{p}}.$$

It follows that

$$\begin{aligned} \widehat{\deg}(\overline{\mathcal{L}} \cdot \overline{x}) &= \sum_{\mathfrak{p} \subseteq \mathcal{O}_K} \operatorname{ord}_{\mathfrak{p}}(s|_{\overline{x}}) \log(n_{\mathfrak{p}}) - \sum_{\sigma \in K(\mathbf{C})} \log \|s(x_\sigma)\|_{can,\sigma} \\ &= \sum_{v \in M_K} (-\log|a_0 x_0 + \ldots + a_n x_n|_v + \log \max |x_i|_v) \\ &= \sum_{v \in M_K} \log \max |x_i|_v \end{aligned}$$

by the product formula. We conclude by dividing both sides by $[K(x) : \mathbf{Q}] = [K : \mathbf{Q}]$.

Just like Weil heights, these Arakelov-theoretic heights satisfy certain functoriality and positivity properties. For instance, it is not hard to prove from the projection formula Proposition 2.2.12 that

Lemma 2.2.23. If $f : \mathcal{X} \to \mathcal{Y}$ is a map of arithmetic varieties over $\mathcal{O}_K$, and $\overline{\mathcal{L}}$ is a Hermitian line bundle on $\mathcal{X}$, then

$$h_{f^*\overline{\mathcal{L}}} = h_{\overline{\mathcal{L}}} \circ f.$$

Lemma 2.2.24. Let $\mathcal{X}/\mathcal{O}_K$ be an arithmetic variety and let $\overline{\mathcal{L}}$ be a Hermitian line bundle on $\mathcal{X}$. Suppose $s \in \widehat{H}^0(\mathcal{X}, \overline{\mathcal{L}})$ is a nonzero effective section of $\overline{\mathcal{L}}$. Then $h_{\overline{\mathcal{L}}}(x) \geq 0$ for all $x \in (\mathcal{X} - |\mathrm{div}(s)|)(\overline{K})$.

Proof. It suffices to prove this for $x \in \mathcal{X}(K)$. If s does not vanish at x, then

$$h_{\overline{\mathcal{L}}}(x) = \widehat{\deg}(\overline{\mathcal{L}} \cdot \overline{x}) = \widehat{\deg}(\widehat{\mathrm{div}}(s|_{\overline{x}})) = \widehat{\deg}(\mathrm{div}(s|_{\overline{x}})) - \sum_{\sigma \in K(\mathbf{C})} \log \|s(x)\|_\sigma .$$

The degree term is certainly nonnegative, and so is the second term by the definition of small section. □

This Lemma 2.2.24 is our first hint that small sections of a line bundle are very important in height theory, which will indeed be the theme of the entire Section 3.

Finally, we make precise the relation between Arakelov-theoretic heights and Weil heights. Note that compared to Weil heights, Arakelov-theoretic heights do not involve choices, of course only after we have chosen arithmetic models.

Theorem 2.2.25. Let X/K be a projective variety and let L be a line bundle on X. For any arithmetic model $(\mathcal{X}, \overline{\mathcal{L}})$ of (X, L) over $\mathcal{O}_K$, $h_{\overline{\mathcal{L}}}$ is a Weil height function associated to L. In other words, $h_{\overline{\mathcal{L}}}$ is in the $O(1)$ class of $\mathbf{h}_L$ as in the notation of Section 2.1.

2.3 Adelic line bundles

From my perspective, there are three major points where the theory of adelic line bundles generalizes the classical Arakelov theory.

(1) First, we allow ourselves to equip metrics not only at the infinite places, but also at the finite places,[14] which come from arithmetic models $(\mathcal{X}, \mathcal{L})$ of (X, L). In particular, everything is done on the projective variety X/K instead of first fixing an arithmetic model. These metrics must satisfy a "coherence condition" reminiscent of the definition of the adèles. For example, in the case of curves, this allows one to transfer Arakelov's admissibility conditions on the curvature of a Hermitian line bundle at the complex

[14] Perhaps keeping with a philosophy that "all places should be treated equally".

places, to the finite places. In [Zha93] this is done using graph theory, and the resulting intersection theory includes information about the places of bad reduction of the curve. Later in [Yua24, Appendix A] this is done using metrics on Berkovich analytic spaces, which is the language that is necessary to extend the results to the relative situation (in Section 2 of loc. cit.).

(2) Second, we would like to be able to take *limits* of line bundles in some suitable sense. For instance, consider an abelian variety A over a number field K and an ample symmetric line bundle L on A. We know that we can define a Néron-Tate height on A via the formula

$$\widehat{h}_L(x) = \lim_{n\to\infty} \frac{1}{4^n} h_L([2]^n(x))$$

where h_L is a Weil height associated to L. The Arakelov height theory as constructed in Section 2.2.5 shows that if $(\mathcal{X}, \mathcal{L})$ is an arithmetic model of (X, L), then $h_{\mathcal{L}}$ differs from h_L, and hence $\widehat{h}_L$, by $O(1)$. We might like to ask we can find such a model where we have equality on the nose. If we could take limits of line bundles, i.e. make sense of "$\lim_{n\to\infty} 4^{-n}(f^n)^*\mathcal{L}$" for some Hermitian line bundle $\mathcal{L}$ on an arithmetic model $\mathcal{A}$ extending (A, L) and $f : \mathcal{A} \to \mathcal{A}$ extending [2], then this could be done. Of course, such a limit does not make sense, but we would very much like it to. In general, this task is impossible unless A has good reduction over K. In that case the Néron model of A is an abelian scheme $\mathcal{A}$, we can extend L to a line bundle $\mathcal{L}$ on $\mathcal{A}$,[15] and [BG06, Corollary 9.5.14, Example 9.5.22] gives the construction of the metrics. On the other hand, in the case of bad reduction, this is known to be impossible.[16]

Let's think about what we need to do actually define such a limit of line bundles. At the infinite places, it is clear what convergence should mean; the Hermitian metrics should simply just (uniformly) converge. On the other hand, it is not as clear what convergence should entail for the underlying line bundles. We may want to look at multiple arithmetic models of (X, L) at once with different underlying spaces $\mathcal{X}$, so we need to figure out how such objects may converge. In keeping with item (1), Zhang's idea is to use the metrics at the finite places (which are induced from the arithmetic models) to track this convergence. In the projective case, this is explained in [Zha95].

(3) Finally, we want to extend the theory from projective arithmetic varieties to quasiprojective ones, which is the subject of the book [YZ24]. This is extremely useful as for situations involving families of varieties, we often need to work with a quasiprojective base (e.g. $\mathcal{C}_g$ over $\mathcal{M}_g$). For instance, even though we can define a Faltings height on $\mathcal{M}_g$, it does not come from a Hermitian line bundle on any compactification of $\mathcal{M}_g$, because the Faltings metric is known to have (albeit mild) logarithmic singularities at

[15] For example, by writing L as a Weil divisor and taking Zariski closures inside $\mathcal{A}$.

[16] I would be very happy if someone could point me to a reference of this fact in the literature.

the boundary (see [YZ24, Sections 2.6.3, 5.5.2] for some explanation). Unfortunately we do not have time or space to explain the ideas in detail, but a good introduction can be found at [Gao25, Chapters 5, 6], or for a very abbreviated summary, [GZ24, Appendix E].

For the rest of these notes, we only require Hermitian metrics to be continuous instead of smooth, so that we match the conventions of [YZ24]. The setup from Section 2.2 can easily be modified to fit this setting.

We now expand our discussion on the above points. Since we can only be very cursory with the constructions due to lack of time and ability of the author, we encourage the reader to also consult the references mentioned below for the full details.

2.3.1 Zhang's classical admissible metrics

In this section we want to review admissible metrics in both the complex and p-adic cases. References include [Yua24, Appendix A.1, A.5], [GY25, Section 5.2], and Zhang's original paper [Zha93].

We begin with a review of Arakelov's admissible metrics in the complex case. Let X be a connected compact Riemann surface of genus $g > 0$. There is a natural Hermitian inner product on the g-dimensional vector space $\Gamma(X, \omega_X)$ given by

$$\alpha \cdot \beta = \frac{i}{2} \int_C \alpha \wedge \overline{\beta}.$$

If $\alpha_1, \ldots, \alpha_g$ is an orthonormal basis with respect to this inner product, we define

Definition 2.3.1. The *Arakelov Kähler form* on X is given by

$$\mu_{Ar} := \frac{i}{2g} \sum_{i=1}^{g} \alpha_i \wedge \overline{\alpha}_i.$$

Moreover, μ_{Ar} is independent of the choice of orthonormal basis.

Definition 2.3.2. We say a smooth Hermitian line bundle $\overline{L}$ on X is *admissible* if $c_1(\overline{L})$ is a scalar multiple of μ_{Ar}. In this case one necessarily has $c_1(\overline{L}) = \deg(L)\mu_{Ar}$.

We also say that a smooth Hermitian line bundle $\overline{M}$ on $X \times X$ is *admissible* if for all $x \in X$, the pullbacks of the metric to $x \times X$ and $X \times x$ are admissible in the above sense.

It is easy to extend the above definition to pairs (D, g_D), where D is a divisor on X and g_D is a Green's function for D, using that such a pair determines a Hermitian line bundle $(\mathcal{O}(D), \exp(-g_D))$.

A reason to introduce admissible metrics is that they are essentially unique for any line bundle, as well as the fact that the given condition means that the metric behaves well with the underlying Kähler form of X (this is very convenient for the intersection theory; see Remark 2.3.7). In particular

Theorem 2.3.3. For any line bundle L on X, there exists an admissible metric of L on X, which is unique up to multiplication by positive constants.

Likewise, for any divisor D on X, there exists an admissible Green's function g_D for D on X, which is unique up to addition by constants. In particular, there is a unique admissible Green's function of D such that $\int_X g_D \mu_{Ar} = 0$. We call it the *strictly admissible Green's function* for D.

We would like to explicitly construct admissible metrics on two natural line bundles, $\mathcal{O}(\Delta) \subseteq X^2$, and ω_X. For each $x \in X$, we have a strictly admissible Green's function $g_x : X - \{x\} \to \mathbf{R}$, which induces a $g_{Ar} : X^2 - \Delta \to \mathbf{R}$ given by $(x, y) \mapsto g_x(y)$. It has the following nice properties:

Theorem 2.3.4. The function $g_{Ar} : X^2 - \Delta \to \mathbf{R}$ given by $(x, y) \mapsto g_x(y)$ is symmetric and is a smooth Green's function for the divisor Δ. Moreover, the metrized Hermitian line bundle $(\mathcal{O}(\Delta), \|\cdot\|_{\Delta,Ar}) \coloneqq (\mathcal{O}(\Delta), \exp(-g_{Ar}))$ on X^2 is admissible.

We can use this construction to induce an admissible metric on ω_X. Since $\omega_C \simeq \mathcal{O}(-\Delta)|_\Delta$ via the residue map, we can define a metric $\|\cdot\|_{Ar}$ by equipping $\mathcal{O}(-\Delta)$ with the dual metric of $\|\cdot\|_{\Delta,Ar}$, and taking the metric on ω_X to be the one such that the residue isomorphism is an isometry. More concretely, the fiber of the residue map at $x \in X$ is induced from the canonical isomorphism

$$(\omega_X \otimes \mathcal{O}(x))|_x \xrightarrow{\sim} \mathbf{C}, \quad t_x^{-1} dt_x \mapsto 1,$$

where t_x is a local coordinate at x. Then unraveling the definitions, $\|\cdot(x)\|_{Ar}$ is the unique metric on the fiber $\omega_X|_x$ such that the above map is an isometry (with the usual absolute value on $\mathbf{C}$), meaning that

$$\|dt_x(x)\|_{Ar} = \lim_{y \to x} |t_x(y)| \exp(g_{Ar}(x, y)).$$

The important theorem is that

Theorem 2.3.5. The Hermitian line bundle $(\omega_C, \|\cdot\|_{Ar})$ is admissible.

In fact, this property uniquely determines the Arakelov Kähler form μ_{Ar} as well.

Having discussed the metrics in the complex setting, we can now arithmetize this construction. Consider the same setup as in Section 2.2.3: K is a number field, $\mathcal{X}$ is a regular arithmetic surface over $\mathcal{O}_K$ with geometrically integral generic fiber $X \coloneqq \mathcal{X}_K$. We also assume that the genus of X is at least 1.

By the above work, for each $\sigma \in K(\mathbf{C})$, we have an Arakelov Kähler form $\mu_{Ar,\sigma}$ on each $\mathcal{X}_\sigma$, which is easily checked to be compatible with complex conjugation $\sigma \mapsto \overline{\sigma}$. Putting everything together, we have

Definition 2.3.6. Let Δ be the diagonal divisor in $\mathcal{X} \times_{\mathcal{O}_K} \mathcal{X}$. Then there is an arithmetic divisor $(\Delta, (g_{Ar,\sigma}))$ on $\mathcal{X} \times_{\mathcal{O}_K} \mathcal{X}$ such that each $g_{Ar,\sigma}$ is admissible (for the Arakelov Kähler form) on $(\mathcal{X} \times_{\mathcal{O}_K} \mathcal{X})_\sigma$. Moreover, if $\omega_{\mathcal{X}/\mathcal{O}_K}$ is the relative dualizing sheaf of $\mathcal{X}/\mathcal{O}_K$ (which is a line bundle by regularity of $\mathcal{X}$), then we call

$$\overline{\omega}_{\mathcal{X}/\mathcal{O}_K,Ar} \coloneqq (\omega_{\mathcal{X}/\mathcal{O}_K}, (\|\cdot\|_{Ar,\sigma}))$$

the *Arakelov canonical sheaf* of $\mathcal{X}/\mathcal{O}_K$. Here $\|\cdot\|_{Ar,\sigma}$ is the admissible metric on $(\omega_{\mathcal{X}/\mathcal{O}_K})_\sigma \cong \omega_{\mathcal{X}_\sigma/\mathbf{C}}$ as constructed above.

The Arakelov canonical sheaf satisfies many useful properties, such an arithmetic adjunction formula. We do not have time to discuss more here, but for more details, see [Mor14, Section 4.5].

Remark 2.3.7. Notice that the strictly admissible condition is very useful for the intersection theory in that if $\overline{\mathcal{D}}_1$ and $\overline{\mathcal{D}}_2$ are arithmetic divisors on $\mathcal{X}$ with strictly admissible Green's functions $(g_{1,\sigma})$ and $(g_{2,\sigma})$, then for all $\sigma \in K(\mathbf{C})$ $\int_{\mathcal{X}_\sigma} g_{2,\sigma} c_1(\mathcal{D}_{1,\sigma}, g_{1,\sigma}) = 0$ by definition. In particular, the integral term in the archimedean contribution to the intersection number (2.2.2) is 0. This gives another reason to introduce admissible metrics.

It is now time to introduce *p-adic metrics* as constructed in [Zha93]. Let K be a non-Archimedean complete discretely valued field, let $\mathcal{O}_K$ be the valuation ring of K, and let κ be its residue field. We agree to normalize the absolute value such that $|\pi| = 1/e$, where $e = 2.718\ldots$. Let X be a smooth projective curve of genus $g > 0$ over K, and assume for simplicity that X has *split semistable reduction* over R.[17] By this, we mean that the minimal regular model $\mathcal{X}$ over $\mathcal{O}_K$ has semistable special fiber, and all the nodes of $\mathcal{X}_\kappa$ are rational over κ. Here, a *semistable curve* C over a field F is a (reduced) projective F-variety of pure dimension 1 such that $C_{\overline{F}}$ is connected and reduced, its singular points are ordinary double points, and any rational irreducible component of C intersects other irreducible components (possibly itself) in at least 2 points. In other words, it is a possibly non-smooth curve with the "nicest possible" singularities.

We now come to an important pair of definitions:

Definition 2.3.8. Let Γ be a *metrized graph*[18] with vertex set $V(\Gamma)$. We say that a *polarized metrized graph* is a pair (Γ, q) where Γ is a metrized graph, $q : V(\Gamma) \to \mathbf{Z}_{\geq 0}$ is a set-theoretic function, and the *canonical divisor*

$$K \coloneqq \sum_{p \in V(\Gamma)} (v(p) - 2 + 2q(p))[p]$$

[17]This assumption is of course not necessary, and can be achieved by taking a finite extension of K by Deligne–Mumford's (semi)stable reduction theorem. But then we need to modify the below definitions to take into account this extension.

[18]See the appendix of [Zha93] or [BR10, Chapter 3] for basics about metrized graphs.

is effective. Here $v(p)$ is the valence of the vertex p.

The terminology in the above definition is taken from [Cin11, Section 4].

Definition 2.3.9. Let X/K be as above, i.e. it is a smooth projective curve with genus $g > 0$, and it has split semistable reduction over R with minimal regular model $\mathcal{X}/\mathcal{O}_K$. We define the *reduction graph* $\Gamma = \Gamma_\kappa = \Gamma(X)$ of X as the *dual graph* of $\mathcal{X}_\kappa$. To be more specific, it is a metrized graph with vertex set indexed by the irreducible components of $\mathcal{X}_\kappa$, edges connecting vertices v, v' (possibly $v = v'$) if and only if the irreducible components of $\mathcal{X}_\kappa$ corresponding to v and v' are joined at a node, and each edge is isometric to the closed unit interval $[0, 1] \subseteq \mathbf{R}$.

Depending on the context, we may also consider the reduction graph Γ as a *polarized metrized graph* as in Definition 2.3.8. In this case, $q : V(\Gamma) \to \mathbf{Z}_{\geq 0}$ is the function that returns the arithmetic genus of the normalized irreducible component corresponding to $v \in V(\Gamma)$.

Here is an easy but important claim. In a sense, it justifies the name "canonical divisor."

Lemma 2.3.10. The canonical divisor $K_{\mathcal{X}_\kappa}$ of Γ_κ has degree $2g - 2$.

Proof. By [Liu02, Proposition 7.5.4],

$$g = g(\mathcal{X}_\kappa) = \sum_{Z \in I(\mathcal{X}_\kappa)} g(Z') + \#(V(\Gamma_\kappa)) - \#I(\mathcal{X}_\kappa) + 1 = \sum_{Z \in I(\mathcal{X}_\kappa)} g(Z') + 1,$$

where g stands for the arithmetic genus, $I(\mathcal{X}_\kappa)$ is the set of irreducible components of $\mathcal{X}_\kappa$, and Z' is the normalization of Z. On the other hand, from the definition of $K_{\mathcal{X}_\kappa}$ one can see that

$$\deg(K_{\mathcal{X}_\kappa}) = \sum_{Z \in I(\mathcal{X}_\kappa)} 2g(Z')$$

because the sum of the valencies of the vertices is equal to twice the number of vertices. □

The idea of the reduction graph is that it is a replacement for the Riemann surface corresponding to X in the archimedean case. In fact, later in Section 2.3.4 we will see that all of the constructions in this section involving reduction graphs are subsumed by more general constructions using Berkovich spaces.

Following Zhang, we can now define a notion of *admissible metrics* on (polarized) metrized graphs Γ, which we will ultimately specialize to the case of reduction graphs. First, let $F(\Gamma)$ be the set of functions $f : \Gamma \to \mathbf{R}$ that are piecewise-smooth and have one-sided derivatives at all points of Γ. Let Δ be a Laplacian operator defined on $f \in F(\Gamma)$ as follows:

$$\Delta f = -f''(x)dx - \sum_{p \in \Gamma} \sum_{v \in T_p(\Gamma)} d_v f(p) \delta_p.$$

To explain the notation, x is a coordinate on each edge of Γ, dx is the Lebesgue measure restricted to each edge, and f'' is only defined away from the (finite) vertex set $V(\Gamma)$. Also, $T_p(\Gamma)$ is the finite set of tangent vectors at p, which has size 2 if $p \notin V(\Gamma)$, and $d_v f(p)$ is the directional derivative of f in the direction $v \in T_p(\Gamma)$ at p. In particular, $\sum_{v \in T_p(\Gamma)} d_v f(p)$ is 0 if $p \notin V(\Gamma)$, so the second sum makes sense.

Now we get an analogue of Green's functions on Γ:

Definition 2.3.11. [Zha93, Section 3.1] Let Γ be a metrized graph with uniform (Lebesgue) metric dx as above. Let μ be a measure on G with mass 1. We say that $g_\mu(x,y) : G \times G \to \mathbf{R}$ is a *Green's function* with respect to μ if it satisfies the following:

(1) g_μ is continuous, piecewise smooth separately in each variable, and symmetric.

(2) For all $x \in \Gamma$,

$$\Delta g_\mu(x, \cdot) = \delta_x - \mu \text{ and } \int g_\mu(x,y)\mu(y) = 0.$$

The Green's function for μ exists and is uniquely defined by these conditions.

The key result [Zha93, Theorem 3.2] is that

Theorem 2.3.12. Let (Γ, q) be a polarized metrized graph with canonical divisor K, which by convention is effective. Then there is a unique metric μ_K of volume 1 on Γ, and a unique constant $c \in \mathbf{R}$, such that for all $x \in \Gamma$,

$$c + g_{\mu_K}(D, x) + g_{\mu_K}(x,x) = 0.$$

Here $g_{\mu_K}(D,x)$ has the obvious definition via extension by linearity. We call this μ_K the *admissible metric on* Γ and g_{μ_K} the *admissible Green's function*.

To relate this to p-adic metrics, we need to make the following definition, which is a baby version of the "model adelic line bundles" we will see in the following sections.

Definition 2.3.13. Let X be a projective variety over a non-Archimedean complete discretely valued field K with algebraic closure $\overline{K}$, and let L be a line bundle on X. We say a *metric* of $L_{\overline{K}}$ is a collection of $\overline{K}$-norms $\|\cdot\|_x$ on closed points $x \in X(\overline{K}) = X_{\overline{K}}(\overline{K})$.

Let F/K be any finite extension, and $\mathcal{O}_F$ the valuation ring of F. Suppose $\mathcal{L}$ is a line bundle on $\mathcal{X}$, where $\mathcal{X}/\mathcal{O}_F$ is a minimal regular integral model for X_F, such that $\mathcal{L}_{\overline{K}} \cong nL_{\overline{K}}$ for some $n \in \mathbf{N}$. In this case, we may associate a natural *model metric* on $L_{\overline{K}}$ as follows: for $x \in X(\overline{K})$, by taking a finite extension of F if necessary, we may assume that $x \in X(F)$ (this does not affect the following construction). By the valuative criterion of properness, x extends to a morphism $\overline{x} : \mathrm{Spec}(\mathcal{O}_F) \to \mathcal{X}$. Then for $l \in x^*L$, we define

$$\|l(x)\| := \inf_{a \in \overline{K}} \{|a|^{1/n} : l \in a\overline{x}^*\mathcal{L}\}.$$

Here $\overline{x}^*\mathcal{L}$ is a 1-dimensional free $\mathcal{O}_F$-module inside the 1-dimensional F-vector space x^*L. This definition does not depend on the choice of F.

These model metrics are induced from (arithmetic/)integral models of (X, L), hence the name.

Definition 2.3.14. Let X be a smooth projective curve over K and let $\omega_{X/K}$ be the canonical sheaf of X. Then we set $\overline{\omega} = \overline{\omega}_{X/K} = (\omega_{X/K}, \|\cdot\|_{Ar})$ to be a metrized line bundle with underlying line bundle $\omega_{X/K}$ and model metrics (induced by relative dualizing sheaves on integral models) on as in the above Definition 2.3.13.

Roughly speaking, for "reasonable" metrized line bundles on X, which are those coming from compactified divisors (D, g) where D is a divisor on X and $g \in F(\Gamma(X))$, there is an intersection theory that takes into account the p-adic metrics/compactifications g, as well as notions of curvature (which will be a measure on $\Gamma(X)$) and Deligne pairing. All of this is developed in [Zha93, Section 2], and for lack of time we cannot write it out here. Granting these constructions, we may say that such a metrized line bundle is *admissible* if its curvature is a multiple of the admissible metric μ_K on $\Gamma(X)$, in line with Definition 2.3.2. We would like to construct an admissible metrized canonical sheaf in this setting. To do so, we need to modify the metric on $\overline{\omega}$ from Definition 2.3.14 via

$$\|\cdot(x)\|_a := \|\cdot(x)\|_{Ar} \cdot \exp(-c - g_{\mu_K}(K, x)). \tag{2.3.1}$$

Here, c and g_{μ_K} are as in Theorem 2.3.12. We need to explain the last term, which requires constructing a map $R : X(\overline{K}) \to \Gamma(X)$. This is explained in [Zha93, Section 2.2], so for simplicity we describe how it works on K-points (since for brevity we did not explain how the construction of the reduction graph works when we base change to extensions of K). Given a minimal regular model $\mathcal{X}$ over $\mathcal{O}_K$ with split semistable reduction, a point $x \in X(K)$ extends to a section $\overline{x} : \mathrm{Spec}(\mathcal{O}_K) \to \mathcal{X}$, which meets the special fiber at a smooth point of irreducible component. Then $R(x)$ is the vertex of $\Gamma(X)$ corresponding to this irreducible component.

To finish, we record the following theorem, which we will need in the proof of Theorem 1.1.1.

Theorem 2.3.15. [Zha93, Theorem 4.4] Let $\omega_a = \omega_{X/K,a}$ denote the metrized line bundle $(\omega_{X/K}, \|\cdot\|_a)$ on X as constructed above. Then with $\langle \cdot, \cdot \rangle$ denoting the Deligne pairing, we have an equality of metrized line bundles on K:

$$\langle \omega_a, \omega_a \rangle = \langle \overline{\omega}, \overline{\omega} \rangle \otimes \mathcal{O}(-\epsilon),$$

where $O(r)$ for a real number r means the metrized line bundle on K whose underlying line bundle is K and $\|1\| = \exp(-r)$. Here, ϵ, which is called *Zhang's ϵ-invariant*, is defined as

$$\int_{\Gamma(X)} g_\mu(x, x)((2g-2)\mu + \delta_K).$$

Moreover, ϵ is nonnegative, and is 0 if and only if $g(X) = 1$ or if X has potentially good reduction.

Remark 2.3.16. It is also possible to define an admissible metric on $\mathcal{O}(\Delta)$, analogously to Definition 2.3.2, where Δ is the diagonal divisor on $X \times X$. For this we refer to [Zha93, Section 4.7].

Remark 2.3.17. Of course, all of the local theory of metrized line bundles developed over non-Archimedean fields above can be patched together to create an analogous theory for global fields, since we already have the theory over the Archimedean fields. This is done in [Zha93, Section 5], but we wait until the next sections to discuss it.

2.3.2 Zhang's classical adelic line bundles on projective varieties

This section is not strictly logically necessary for the theory of adelic line bundles over quasiprojective varieties, but it is useful to build intuition in a more concrete case. Therefore, we will be very brief. We will follow the original paper [Zha95].

For the rest of this section, let K be a number field, and let M_K be the set of places of K. Let X be a *projective* variety over K and let L be a line bundle on X.

Definition 2.3.18. An *adelic metric* on L is a collection of continuous bounded K_v-metrics $(\|\cdot\|_v)_v$ of L_{K_v} on X_{K_v}, for each $v \in M_K$, such that the following *coherence condition* is satisfied:

There is a nonempty open subset $U \subseteq \operatorname{Spec}(\mathcal{O}_K)$, a projective flat variety $\mathcal{X}$ on U with generic fiber X, a line bundle $\mathcal{L}$ on $\mathcal{X}$ extending L, such that for all closed points $v \in U$, the K_v-metric $\|\cdot\|_v$ is induced by the model $(\mathcal{X} \times_U \mathcal{O}_{K_v}, \mathcal{L} \times_U \mathcal{O}_{K_v})$ as in Definition 2.3.13.

We call $\overline{L} := (L, (\|\cdot\|_v)_v)$ an *adelic line bundle* on X. The above condition should justify the adjective "adelic."

Definition 2.3.19. A metric on $\mathcal{L}$ is *continous* (resp. *bounded*) if there is a projective model $(\mathcal{X}, \mathcal{M})$ such that $\|\cdot\| / \|\cdot\|_{\mathcal{M}}$, as a well-defined function on $\overline{K}$-points, is continuous (resp. *bounded*) in the topology on $X(\overline{K})$ induced from the topology on $\overline{K}$.

Example 2.3.20. Suppose (X, L) has an arithmetic model $(\mathcal{X}, \mathcal{L})$, where we only require that $\mathcal{L}$ restricts to a positive multiple of L on the generic fiber (**from now on we will use "arithmetic model" in this generalized sense**), and assume for convenience that $\mathcal{X}$ is also regular. Then by the construction in Definition 2.3.13, the arithmetic model induces a collection of K_v-metrics on L which is continuous and bounded. It turns out that this collection is also an adelic metric on L.

Such an adelic line bundle L is called a *model adelic line bundle*, in the sense that all the metrics come from a single arithmetic model of (X, L) over $\mathcal{O}_K$.

The most useful innovation of adelic line bundles is the ability to take limits. The limiting process is defined and checked on these adelic metrics, but it is perhaps more intuitive to think of the limiting process as happening in arithmetic models of (X, L). In fact this is what will be generalized in the theory of adelic line bundles on *quasiprojective* varieties in the next section.

Definition 2.3.21. We say that a sequence $\|\cdot\|_n$ of adelic metrics on L *converges* to an adelic metric $\|\cdot\|$ (the limit) if there is an open subset U of $\mathrm{Spec}(\mathcal{O}_K)$ such that for each $\mathfrak{p} \in U$, $\|\cdot\|_{n,\mathfrak{p}} = \|\cdot\|_{\mathfrak{p}}$ for all n, and $\|\cdot\|_{n,\mathfrak{p}} / \|\cdot\|$ converges to 1 uniformly on $X(\overline{K}_{\mathfrak{p}})$ for all $\mathfrak{p}$.

In this way, we may talk about adelic line bundles that might not necessarily come from single arithmetic models, but rather as the limit of a sequence of such models.

We now want to set up the top intersection number of adelic line bundles. It will be defined as a limit from arithmetic intersection numbers of models. However, because our metrics are only continuous now (most glaringly, there is no notion of smoothness at the p-adic places), we need to impose an extra integrability condition on the metrics.

Definition 2.3.22. We say a model adelic line bundle (X, L) induced by an arithmetic model $(\mathcal{X}, \overline{\mathcal{L}})$ is *nef* if the Hermitian line bundle $\mathcal{L}$ is nef, in the sense that it has nonnegative degree on any curve contained in a special fiber, and the curvature form of $\mathcal{L}_{\mathbf{C}}$ on the complex manifold $X(\mathbf{C})$ is semipositive. An adelic line bundle on X is nef if it is isometric to the limit of a sequence of nef model adelic line bundles on X. We also say an adelic line bundle on X is *integrable* if it is isometric to the difference of two nef adelic line bundles.

We now have the definition–theorem

Theorem 2.3.23. [Zha95, Theorem 1.4] Let $\overline{L}_1, \ldots, \overline{L}_{d+1}$ be nef adelic line bundles on X, where $d = \dim(X)$. Assume that $\|\cdot\|_i$, $1 \leq i \leq d+1$, is the limit of model adelic metrics induced by projective models $(\mathcal{X}_{i,n}, \overline{\mathcal{L}}_{i,n})$ with the $\mathcal{L}_{i,n}$ nef and $\mathcal{L}_{i,n}|_X = e_{i,n}\mathcal{L}_i$. Then

$$\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{d+1} \coloneqq \lim_{n_1,\ldots,n_{d+1}\to\infty} \frac{1}{e_{1,n_1} \cdots e_{d+1,n_{d+1}}} \overline{\mathcal{L}}_{1,n_1} \cdots \overline{\mathcal{L}}_{d+1,n_{d+1}}$$

exists and does not depend on the $(\mathcal{X}_{i,n}, \overline{\mathcal{L}}_{i,n})$. Here by abuse of notation, the right-hand side is the intersection number of the pullbacks of the $\overline{\mathcal{L}}_{i,n_i}$ to a common integral model $\mathcal{X}$ dominating the $\mathcal{X}_{i,n_i}$.[19] This defines a symmetric multilinear intersection product, and using the multilinearity it can be extended to integrable adelic line bundles.

With the top intersection number in place, we can define heights of algebraic points exactly as in Section 2.2.5, taking into account the limit process. This is a special case of [Zha95, Definition 1.9], which in fact defines the height of closed subvarieties of X.

[19] This $\mathcal{X}$ can be constructed by taking the Zariski closure of X^{d+1} inside the fiber product of the $\mathcal{X}_{i,n_i}$'s over $\mathcal{O}_K$.

Definition 2.3.24. Let $\overline{L}$ be an integrable adelic line bundle and let $x \in X(\overline{K})$. The *height* of x with respect to $\overline{L}$ is given by

$$h_{\overline{L}}(x) \coloneqq \frac{\deg(\overline{L}|_x)}{[K(x):\mathbf{Q}]}.$$

Above, the numerator of the right-hand side is the "0-dimensional" intersection number, which is calculated by the limit of the arithmetic degrees of arithmetic models of $(x, \overline{L}|_x)$. We include the deg to lessen possible confusion.

Example 2.3.25. Assume for this example that $\overline{L}$ is in fact a model adelic line bundle induced by $(\mathcal{X}, \mathcal{L})$. Then the Zariski closure of x is some $\operatorname{Spec}(\mathcal{O}_F)$ for a finite extension $F = K(x)$ of K. Then $h_{\overline{L}}(x)$ is equal to $\widehat{\deg}(\mathcal{L}|_{\overline{x}})/[F:\mathbf{Q}]$. For a nonzero rational section s of L whose support does not contain x, it extends to a section $\overline{s}$ of $\mathcal{L}$, and we calculate

$$\begin{aligned}
\widehat{\deg}(\mathcal{L}|_{\overline{x}}) &= \widehat{\deg}\left(\operatorname{div}(\overline{s}|_{\overline{x}}), -\sum_{\sigma \in F(\mathbf{C})} \log \|s(x)\| \, [\sigma]\right) \\
&= \sum_{\mathfrak{p} \in \operatorname{maxSpec}(\mathcal{O}_F)} \operatorname{ord}_{\mathfrak{p}}(\overline{s}|_{\overline{x}}) \log(N\mathfrak{p}) - \sum_{\sigma \in F(\mathbf{C})} \log \|s(x)\|_\sigma \\
&= -\sum_{\mathfrak{p} \in \operatorname{maxSpec}(\mathcal{O}_F)} \log \|s(x)\|_{\mathfrak{p}} - \sum_{\sigma \in F(\mathbf{C})} \log \|s(x)\|_\sigma \\
&= -\sum_{v \in M_F} \log \|s(x)\|_v^{e_v}.
\end{aligned}$$

Note that by Theorem 2.2.25,

$$h_{\overline{\mathcal{L}}}(x) = -\frac{1}{[F:\mathbf{Q}]} \sum_{v \in M_F} \log \|s(x)\|_v^{e_v}$$

is in fact a Weil height for X associated to the line bundle L.

The crucial application of this limiting process is the construction of an adelic line bundle inducing the Néron-Tate height. Recall that we previously mentioned, in item (2) of the overview to this Section 2.3, that the Néron-Tate height of an abelian variety A with respect to a symmetric ample line bundle L may not come from the Arakelov-theoretic height on single arithmetic model $(\mathcal{A}, \mathcal{L})$. It turns out that we have now fixed this problem: by our definition of the limiting process, we can give a meaning to "$\lim_{n\to\infty} 4^{-n}(f^n)^*\mathcal{L}$," where $f : \mathcal{A} \to \mathcal{A}$ extends $[2] : A \to A$. We sketch the construction but refer to [Zha95, Section 2.1–2.3] for details.

Example 2.3.26. Given (A, L), pick any arithmetic model $(\mathcal{A}, \mathcal{L})$ (the Hermitian metric does not matter). Then $[2]$ extends to a *rational* map $\mathcal{A} \dashrightarrow \mathcal{A}$, and upon blowing up the indeterminacy locus, we get an actual morphism $f_1 : \mathcal{A}_1 \to \mathcal{A}$ for some integral model $\mathcal{A}_1$ of A extending $[2]$ on the generic fiber. We can pull back $\mathcal{L}$ via f_1 to get a Hermitian line bundle $\mathcal{L}_1$ on $\mathcal{A}_1$ extending $4L$ on the generic fiber.[20] Repeating this process for $[2]^n$ for all $n \in \mathbf{N}$, we get a sequence of arithmetic models $(\mathcal{A}_n, \mathcal{L}_n)$ of (A, L). The model metrics induced by these models converge to an adelic metric $\|\cdot\|$ on L, and the height of this *adelic line bundle* gives the Néron-Tate height associated to $\mathcal{L}$, using the uniqueness of the Néron-Tate height as the only quadratic function in its equivalence class modulo bounded functions.

Remark 2.3.27. In fact this construction can be significantly generalized to the setting of arithmetic dynamics. We say that a *polarized dynamical systems* consists of the following data: a projective variety X over a number field K, a self-map $f : X \to X$ such that there exists an ample line bundle L on X with $f^*L = dL$ for some $d \geq 2$. An example is $X = \mathbf{P}^n$, f a polynomial map of degree d, $L = \mathcal{O}(1)$. Another example is X an abelian variety, f the multiplication by $n \geq 2$ map, and L any symmetric ample line bundle. Then using the Tate limit argument, one can define the (Call–Silverman) *canonical height* associated to this dynamical system from a Weil height associated to L. Essentially the same argument as in Example 2.3.26 shows that this canonical height is induced from an adelic line bundle on X with underlying line bundle L.

2.3.3 Adelic line bundles on quasiprojective varieties

It is time to do Yuan–Zhang's theory of adelic line bundles over quasiprojective varieties, which generalizes the constructions from Section 2.3.2. The main difficulty is that even the basic constructions involve many pieces and can be quite complicated to parse, and likewise there is a lot of notation to set up. This complication is compounded by the fact that the theory aims to create a uniform framework for both number fields and function fields, so in order to unify these cases there are frequent abuses of notation, which can be confusing when first learning the material. To follow [YZ24], we will keep these abuses of notation, but try to clarify what is going on at certain intervals. The main reference for this material is of course the book [YZ24]. Shorter summaries, which cover most of what we need below, can be found at [Gao25, Chapters 5, 6] and [GZ24, Appendix E].

There are two parallel cases that we will work in. In the *arithmetic case*, we set $k = \mathbf{Z}$, and in the *geometric case*, k is a field. We have the following table of "*uniform notation*", which will be expanded as we go. The general rule for reading this "uniform notation" is that notations and definitions in the geometric setting are simply those in the arithmetic setting, minus any mention of "compactifications" (e.g. Green's functions, Hermitian metrics, etc.).

[20] This is a good reason for why in the Definition 2.3.13 of model metrics, we want to allow arithmetic models to restrict to *multiples* of L on the generic fiber, and not insist that they restrict to exactly L.

	Arithmetic case, $k = \mathbf{Z}$	Geometric case, k a field
Finitely generated field K/k	Finitely generated field over $\mathbf{Q}$	Finitely generated field over k
Projective variety $\mathcal{X}/k$	$\mathcal{X}$ is an arithmetic variety over $\mathbf{Z}$ as in Definition 2.2.1.	Usual meaning, and we require the variety to be integral.
Quasiprojective variety $\mathcal{U}/k$	$\mathcal{U}$ is an open subscheme of an arithmetic variety over $\mathbf{Z}$.	Usual meaning
Projective model $\mathcal{X}/k$ of a quasiprojective variety $\mathcal{U}/k$	$\mathcal{X}/k$ is a projective variety and $\mathcal{U} \hookrightarrow \mathcal{X}$ is an open immersion over k.	Same as in the arithmetic case
$\widehat{\mathrm{Div}}(\mathcal{X})$ where $\mathcal{X}$ is a projective variety over k	As defined in Section 2.2.1	$\mathrm{Div}(\mathcal{X})$
$\widehat{\mathrm{Pr}}(\mathcal{X})$ where $\mathcal{X}$ is a projective variety over k	As defined in Definition 2.2.8	Principal (Cartier) divisors on $\mathcal{X}$
$\widehat{\mathrm{Pic}}(\mathcal{X})$ where $\mathcal{X}$ is a projective variety over k	As defined in Section 2.2.1	$\mathrm{Pic}(\mathcal{X})$
$\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})$ where $\mathcal{X}$ is a projective variety over k	Groupoid of Hermitian line bundles on $\mathcal{X}$	Groupoid of line bundles on $\mathcal{X}$
$\widehat{\mathrm{div}}(s)$ where s is a rational section of $\mathcal{L} \in \widehat{\mathrm{Pic}}(\mathcal{X})$	As defined in Section 2.2.1	Usual meaning

Table 1: Some notations for [YZ24]

Although k is always $\mathbf{Z}$ or a field, we ultimately care about (projective) varieties over number fields (resp. function fields over k), which are not projective or quasiprojective over $k = \mathbf{Z}$ (resp. k a field) since they are not finite type. This motivates the following strange-looking definition:

Definition 2.3.28. Let $k = \mathbf{Z}$ or a field. An *essentially quasiprojective variety* over k is a quasiprojective (integral) variety X over a finitely generated field K/k, or in the case $k = \mathbf{Z}$, we allow $K = \mathbf{Z}$ as well.[21] A *(quasi)projective model* of an essentially quasiprojective variety X is a (quasi)projective variety $\mathcal{U}/k$ along with a map of k-schemes $i : X \to \mathcal{U}$ satisfying the following conditions:

(1) The map i is injective on underlying sets.

(2) All of the maps of stalks $\mathcal{O}_{\mathcal{U},i(x)} \to \mathcal{O}_{X,x}$ induced by i are isomorphisms.

The conditions on i seem strange, but in fact they imply a reasonable intuition that X "comes from" the generic fiber of a morphism of quasiprojective varieties over k. More formally, we have from [YZ24, Lemma 2.3.3]

Lemma 2.3.29. Let K/k be a finitely generated field, X a quasiprojective variety over K, and $i : X \to \mathcal{U}$ a quasiprojective model of X over k. Then there is an open subscheme $\mathcal{U}'$ of $\mathcal{U}$ containing the image $i(X)$, along with a flat morphism $j : \mathcal{U}' \to \mathcal{V}$ of quasiprojective varieties over k, such that the function field $K(\mathcal{V}) \cong K$ and the generic fiber of j is isomorphic to $X \to \operatorname{Spec}(K)$.

This lemma shows that the notions of essentially quasiprojective variety and projective models thereof are not foreign at all—for instance, the integral models as introduced in Definition 2.2.21 are examples.

Next, recall in Section 2.3.2 that we considered (slightly generalized) arithmetic models of projective varieties, where the generic fiber of the extended line bundle on the integral model was allowed to be a positive integer multiple of the original line bundle. Namely, for a line bundle L on a (essentially) quasiprojective variety X/k, we would like to take a (quasi)projective model $\mathcal{X}$ of X along with a line bundle $\mathcal{L}$ on $\mathcal{X}$ whose restriction to X is an integral multiple nL. In other words, the $\mathbf{Q}$-line bundle $(1/n)\mathcal{L}$ restricts to L. This motivates the introduction of $\mathbf{Q}$-coefficients for divisors and line bundles.

Definition 2.3.30. Let $\mathcal{X}$ be a projective variety over k. We denote by $\widehat{\operatorname{Div}}(\mathcal{X})_{\mathbf{Q}}$ the tensor product $\widehat{\operatorname{Div}}(\mathcal{X}) \otimes_{\mathbf{Z}} \mathbf{Q}$, and similarly for $\widehat{\operatorname{Pic}}(\mathcal{X})_{\mathbf{Q}}$. We say a $\mathbf{Q}$-divisor $\mathcal{D} \in \widehat{\operatorname{Div}}(\mathcal{X})_{\mathbf{Q}}$ is

[21] This is only a special case of the definition as in [YZ24, Section 2.3.2], but it is the only case we need for our exposition. We keep the terminology "essentially quasiprojective variety" as it lessens possible confusion between quasiprojective varieties over K and quasiprojective varieties over k, and we also don't need to keep specifying the field K.

effective (resp. *nef*) if there exists $n \in \mathbf{N}$ such that $n\mathcal{D}$ is an effective (resp. nef) integral divisor in $\widehat{\mathrm{Div}}(\mathcal{X})$. In the arithmetic case, we have a notion of strictly effective $\mathbf{Q}$-divisor in the same way.

Let $q\mathcal{L} \in \widehat{\mathrm{Pic}}(\mathcal{X})_{\mathbf{Q}}$ be a $\mathbf{Q}$-line bundle. A *global section* of $q\mathcal{L}$ is an element of $\varinjlim_m \Gamma(\mathcal{X}, qm\mathcal{L})$, where m runs through positive integers such that $qm \in \mathbf{Z}$. Here the direct limit has transition maps given by $m|n$, sending $s \in \Gamma(\mathcal{X}, qm\mathcal{L})$ to the appropriate tensor power in $\Gamma(\mathcal{X}, qn\mathcal{L})$. Similarly, a *rational section* of $q\mathcal{L}$ is an element of $\varinjlim_m \Gamma(\eta, qm\mathcal{L}_\eta)$, where η is the generic point of $\mathcal{X}$.

Finally, for s a rational section of $q\mathcal{L}$ represented by $s_m \in \Gamma(\eta, qm\mathcal{L}_\eta)$, we define $\widehat{\mathrm{div}}(s) := \widehat{\mathrm{div}}(s_m)/m$, which is a $\mathbf{Q}$-divisor on $\mathcal{X}$.

Definition 2.3.31. Let $k = \mathbf{Z}$ or a field, and let $\mathcal{U}$ be an *open* subscheme of a projective variety $\mathcal{X}$ over k. We define $\widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U})$ to be the fiber product of abelian groups

$$\widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U}) := \widehat{\mathrm{Div}}(\mathcal{X})_{\mathbf{Q}} \times_{\widehat{\mathrm{Div}}(\mathcal{U})_{\mathbf{Q}}} \widehat{\mathrm{Div}}(\mathcal{U}).$$

In other words, it is the group of pairs $(\mathcal{D}, \mathcal{D}')$ where $\mathcal{D} \in \mathrm{Div}(\mathcal{X})_{\mathbf{Q}}$, $\mathcal{D}' \in \mathrm{Div}(\mathcal{U})$, and their natural restrictions to $\mathrm{Div}(\mathcal{U})_{\mathbf{Q}}$ agree. Note that there is a natural map $\widehat{\mathrm{Div}}(\mathcal{X}) \to \widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U})$ given by $\mathcal{D} \mapsto (\mathcal{D}, \mathcal{D}|_{\mathcal{U}})$.

We say that an element $(\mathcal{D}, \mathcal{D}') \in \widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U})$ is *effective* if its both components are effective in their respective groups. We say it is *nef* if $\mathcal{D}$ is nef. Note that effectivity induces a partial order, denoted $\geq$, on elements of $\widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U})$.

We have set up enough notation to define adelic divisors. As before, we first have a notion of model adelic divisors, which are those coming from a *single* projective model. To begin, we work in the quasiprojective case.

Definition 2.3.32 (Model adelic divisors)**.** Let k be $\mathbf{Z}$ or a field, and let $\mathcal{U}$ be a quasiprojective variety over k. By pullback of divisors, the groups $\widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U})$ form a directed system as $\mathcal{X}$ runs over all projective models of $\mathcal{U}$. We define

$$\widehat{\mathrm{Div}}(\mathcal{U}/k)_{mod} := \varinjlim_{\mathcal{X}} \widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U}), \quad \widehat{\mathrm{Pr}}(\mathcal{U}/k)_{mod} := \varinjlim_{\mathcal{X}} \widehat{\mathrm{Pr}}(\mathcal{X}).$$

This defines model adelic divisors for a quasiprojective variety $\mathcal{U}/k$, and it remains to introduce the limiting process. As in Section 2.3.2, we need a way to measure convergence of model adelic divisors on different projective models of $\mathcal{U}$. This is achieved by equipping $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{mod}$ with a certain metric and completing with respect to this metric.

The definition of effectivity as in Definition 2.3.31 extends to $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{mod}$ in the evident way, and in particular the partial order $\geq$ as introduced in Definition 2.3.31 extends to a partial order on $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{mod}$. We define a *boundary divisor* of $\mathcal{U}/k$ (still quasiprojective) to be a pair $(\mathcal{X}_0, \overline{\mathcal{E}}_0)$ consisting of a projective model $\mathcal{X}_0$ of $\mathcal{U}$ and an *strictly* effective Cartier divisor $\overline{\mathcal{E}}_0 \in \mathrm{Div}(\mathcal{X}_0)$ with the support of the underlying divisor $\mathcal{E}_0$ equal to $\mathcal{X}_0 - \mathcal{U}$. Note that in the geometric case, there is no Green's function, so "strictly effective" just means effective.

Definition 2.3.33 (Boundary norm)**.** Let $\mathcal{U}/k$ be a quasiprojective variety and $(\mathcal{X}_0, \overline{\mathcal{E}}_0)$ a boundary divisor. We define the *boundary norm* on the group $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{mod}$ with respect to this data to be

$$\|\cdot\|_{\overline{\mathcal{E}}_0} : \widehat{\mathrm{Div}}(\mathcal{U}/k)_{mod} \to [0, \infty], \quad \left\|\overline{\mathcal{D}}\right\|_{\overline{\mathcal{E}}_0} := \inf\{\epsilon \in \mathbf{Q}_{>0} : -\epsilon\overline{\mathcal{E}}_0 \leq \overline{\mathcal{D}} \leq \epsilon\overline{\mathcal{E}}_0\}.$$

Note that the value ∞ is possible (in the case that of the infinimum of an empty set). It induces a *boundary topology* on $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{mod}$.

It turns out [YZ24, Lemma 2.4.1] that $\|\cdot\|_{\overline{\mathcal{E}}_0}$ vanishes if and only if $\overline{\mathcal{D}} = 0$, and it satisfies the triangle inequality. Most importantly, although the boundary norm depends on the choice of $(\mathcal{X}_0, \overline{\mathcal{E}}_0)$, any two choices give equivalent norms in the usual sense. Therefore the boundary topology does not depend on the choice of boundary divisor, and neither does the following definition:

Definition 2.3.34. [Adelic divisors, quasiprojective case] Let $\mathcal{U}/k$ be a quasiprojective variety. The group of *adelic divisors* of $\mathcal{U}$, $\widehat{\mathrm{Div}}(\mathcal{U}/k)$, is defined to be the completion of $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{mod}$ with respect to $\|\cdot\|_{\overline{\mathcal{E}}_0}$ for any choice of boundary divisor $(\mathcal{X}_0, \overline{\mathcal{E}}_0)$.

We also define $\widehat{\mathrm{CaCl}}(\mathcal{U}/k)$ to be the quotient $\widehat{\mathrm{Div}}(\mathcal{U}/k)/\widehat{\mathrm{Pr}}(\mathcal{U}/k)_{mod}$. It turns out [YZ24, Lemma 2.4.3] that this group is canonically isomorphic to the other reasonable definition of $\widehat{\mathrm{CaCl}}(\mathcal{U}/k)$, which is the completion of $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{mod}/\widehat{\mathrm{Pr}}(\mathcal{U}/k)_{mod}$ with respect to the (induced) boundary topology.

In particular, we represent elements of $\widehat{\mathrm{Div}}(\mathcal{U}/k)$ by Cauchy sequences whose terms are model adelic divisors. Finally:

Definition 2.3.35. [Adelic divisors, essentially quasiprojective case] Let X/k be an essentially quasiprojective variety. Then groups $\widehat{\mathrm{Div}}(\mathcal{U}/k)$ form a directed system via pullback as $\mathcal{U}$ runs over quasiprojective models $\mathcal{U}/k$ of X. We define $\widehat{\mathrm{Div}}(X/k)$ as the direct limit $\varinjlim_{\mathcal{U}} \widehat{\mathrm{Div}}(\mathcal{U}/k)$, and likewise for $\widehat{\mathrm{CaCl}}(X/k)$.

Note that if X is actually a quasiprojective variety over k, then X is a quasiprojective model of itself, and so this definition is compatible with the previous Definition 2.3.34.

We now turn to the line bundle side of the theory. As with adelic divisors, we will need to take limits, but here we will define the relevant limits in a single step.

To start, we work with a quasiprojective variety $\mathcal{U}/k$, and we need a way to compare line bundles on different projective models of $\mathcal{U}$. Let $\mathcal{X}_1, \mathcal{X}_2$ be projective models of $\mathcal{U}$ and let $\overline{\mathcal{L}}_i \in \widehat{\mathrm{Pic}}(\mathcal{X}_i)_{\mathbf{Q}}$. By taking a diagonal embedding we can always find a projective model $\mathcal{Y}/k$ of $\mathcal{U}$ with morphisms $\tau_i : \mathcal{Y} \to \mathcal{X}_i$ over k. Suppose $l : \overline{\mathcal{L}}_1 \dashrightarrow \overline{\mathcal{L}}_2$ is a *rational map*, meaning that l is an isomorphism $\mathcal{L}_1|_U \to \mathcal{L}_2|_U$ of underlying $\mathbf{Q}$-line bundles restricted to $\mathcal{U}$.[22] Then l is a rational section of the underlying line bundle of $\tau_1^*\overline{\mathcal{L}}_1^\vee \otimes \tau_2^*\overline{\mathcal{L}}_2$ on $\mathcal{Y}$, and so it defines an arithmetic $\mathbf{Q}$-divisor $\widehat{\mathrm{div}}_{\mathcal{Y}}(l)$ on $\mathcal{Y}$, which can be considered as a model adelic divisor $\widehat{\mathrm{div}}(l) \in \widehat{\mathrm{Div}}(\mathcal{U}/k)_{mod}$. Note that the component of $\widehat{\mathrm{div}}(l)$ in $\widehat{\mathrm{Div}}(\mathcal{U})$ is 0.

Let us fix a boundary divisor $(\mathcal{X}_0, \overline{\mathcal{E}}_0)$, and hence a boundary topology on $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{mod}$. Then we have the key definition

Definition 2.3.36. [Adelic line bundles, quasiprojective case] Let $\mathcal{U}/k$ be a quasiprojective variety. An object of the *category* $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$ of adelic line bundles given by a tuple $\overline{\mathcal{L}} := (\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, l_i)_{i\geq 1})$ as follows:

(1) $\mathcal{L}$ is a (vanilla) line bundle on $\mathcal{U}$. We call it the *underlying line bundle* of the object.

(2) The $\mathcal{X}_i$'s are projective models of $\mathcal{U}$ over k.

(3) $\overline{\mathcal{L}}_i$ is an element of $\widehat{\mathrm{Pic}}(\mathcal{X}_i)_{\mathbf{Q}}$.

(4) $l_i : \mathcal{L} \to \mathcal{L}_i|_{\mathcal{U}}$ is an isomorphism of ($\mathbf{Q}$)-line bundles over $\mathcal{U}$.

(5) The sequence is *Cauchy* in the following sense. From item (4), each $l_i l_1^{-1}$ is an isomorphism $\mathcal{L}_1|_U \to \mathcal{L}_i|_U$ of $\mathbf{Q}$-line bundles, so is a rational map $\overline{\mathcal{L}}_1 \dashrightarrow \overline{\mathcal{L}}_i$. By the above discussion, we have model adelic divisors $\widehat{\mathrm{div}}(l_i l_1^{-1}) \in \widehat{\mathrm{Div}}(\mathcal{U}/k)_{mod}$. Then this sequence of model adelic divisors must be a Cauchy sequence under the boundary topology.

Sometimes, by abuse of language, we call the tuple $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, l_i)_{i\geq 1})$ a "Cauchy sequence."

In other words, an adelic line bundle with underlying line bundle $\mathcal{L}$ is literally a sequence of (arithmetic) projective models of the pair $(\mathcal{U}, \mathcal{L})$, with the "convergence data" of these models tacked on to the object.

There are notions of isomorphism, tensor product, and dual of adelic line bundles, which are defined "termwise" in the terms of the tuple giving an adelic line bundle. For example, the dual of $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, l_i))$ is given by $(\overline{\mathcal{L}}^\vee, (\mathcal{X}_i, \overline{\mathcal{L}}_i^\vee, l_i^\vee))$, and the tensor product of $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, l_i))$ and $(\mathcal{L}', (\mathcal{X}'_i, \overline{\mathcal{L}}'_i, l'_i))$ is given by

$$(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, l_i)) \otimes (\mathcal{L}', (\mathcal{X}'_i, \overline{\mathcal{L}}'_i, l'_i)) := (\mathcal{L} \otimes \mathcal{L}', (\mathcal{W}_i, \tau_i^*\overline{\mathcal{L}}_i \otimes (\tau'_i)^*\overline{\mathcal{L}}'_i, l_i \otimes l'_i))$$

[22] See [YZ24, Section 2.2.1] for the definition of an isomorphism of $\mathbf{Q}$-line bundles.

where $\mathcal{W}_i$ is any projective model of $\mathcal{U}$ equipped with maps $\tau_i : \mathcal{W}_i \to \mathcal{X}_i$, $\tau_i' : \mathcal{W}_i \to \mathcal{X}_i'$. We refer to [YZ24, Section 2.5.1] for more details.

In particular, we may define $\widehat{\mathrm{Pic}}(U/k)$ as the *group* of isomorphism classes of adelic line bundles. The identity element is $(\mathcal{O}_U, (\mathcal{X}_0, \overline{\mathcal{O}}_{\mathcal{X}_0}, 1))$, a constant sequence. By unraveling the definitions, one can show that

Proposition 2.3.37. If $\mathcal{U}/k$ is a quasiprojective variety, then there is a canonical isomorphism $\widehat{\mathrm{CaCl}}(\mathcal{U}/k) \xrightarrow{\sim} \widehat{\mathrm{Pic}}(\mathcal{U}/k)$.

The proof is given in [YZ24, Proposition 2.5.1], and the idea is to simply apply the canonical functor $\mathcal{O}(\cdot)$ from (arithmetic) divisors to (Hermitian) line bundles on a representing Cauchy sequence of an element in $\widehat{\mathrm{CaCl}}(\mathcal{U}/k)$. This will give the "tuple" data of an adelic line bundle.

To get the definitions for essentially quasiprojective varieties X/k, we do the same thing as in Definition 2.3.35.

Definition 2.3.38. [Adelic line bundles, essentially quasiprojective case] Let X/k be an essentially quasiprojective variety. Then groups $\widehat{\mathrm{Pic}}(\mathcal{U}/k)$ form a directed system via pullback as $\mathcal{U}$ runs over quasiprojective models $\mathcal{U}/k$ of X. We define $\widehat{\mathrm{Pic}}(X/k)$ as the direct limit $\varinjlim_{\mathcal{U}} \widehat{\mathrm{Pic}}(\mathcal{U}/k)$.

By definition, Proposition 2.3.37 holds true in the essentially quasiprojective case as well.

Example 2.3.39. At this point it is an interesting exercise to compute $\widehat{\mathrm{Div}}(\mathrm{Spec}(K)/\mathbf{Z})$ for a number field K. This is worked out in detail in [YZ24, Lemma 2.6.1], but the answer is that

$$\widehat{\mathrm{Div}}(\mathrm{Spec}(K)/\mathbf{Z}) \cong \varinjlim_{\mathcal{U}} \left(\bigoplus_{v \in |\mathcal{U}|} \mathbf{Z} \oplus \bigoplus_{v \in |\mathrm{Spec}(\mathcal{O}_K) - \mathcal{U}| \cup M_{K,\infty}} \mathbf{R} \right),$$

where $\mathcal{U}$ runs over nonempty open subschemes of $\mathrm{Spec}(\mathcal{O}_K)$, and $|\mathcal{U}|$ denotes the places of K corresponding to the closed points of $\mathcal{U}$.

Before we move on to the intersection theory of adelic line bundles and divisors, we need some positivity and functoriality properties of adelic line bundles.

Definition 2.3.40. Let $\mathcal{U}/k$ be a quasiprojective variety and let $\overline{\mathcal{L}} = (\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, l_i)_{i \geq 1})$ be an adelic line bundle on $\mathcal{U}/k$.

(1) If each $\overline{\mathcal{L}}_i$ is nef on $\mathcal{X}_i$, we call $\overline{\mathcal{L}}$ *strongly nef.*

(2) If there exists a strongly nef adelic line bundle $\overline{\mathcal{L}}'$ such that $a\overline{\mathcal{L}} + \overline{\mathcal{L}}'$ is strongly nef for all $a \in \mathbf{N}$, then we call $\overline{\mathcal{L}}$ *nef.*

(3) If $\overline{\mathcal{L}}$ is isomorphic to the difference of two strongly nef adelic line bundles, we call it *integrable.*

If X/k is essentially quasiprojective, then we say that $\overline{\mathcal{L}} \in \widehat{\mathrm{Pic}}(X/k)$ is strongly nef (resp. nef, integrable) if it has can be represented by an element in some $\widehat{\mathrm{Pic}}(\mathcal{U}/k)$ with that property, where $\mathcal{U}$ is a quasiprojective model of X.

We can transfer these notions to adelic divisors in the evident way. If we want to denote the *subsets* of $\widehat{\mathrm{Pic}}(\mathcal{U}/k)$ consisting of strongly nef (resp. nef, integrable) elements, we will append the appropriate subscript "snef" (resp. "nef," "int").

Example 2.3.41. Another application that we can mention at this point is to construct invariant line bundles for *(relative) polarized dynamical systems* (X, f, L) over S, where S is an essentially quasiprojective variety over k, X is a flat projective S-scheme, $f : X \to X$ is an S-morphism, and $L \in \mathrm{Pic}(X)_{\mathbf{Q}}$ is a relatively ample $\mathbf{Q}$-line bundle such that $f^*L = qL$ for some $q \in \mathbf{Q}_{>1}$. Then one can use the Tate limit process to construct an invariant line bundle for this dynamical system, which is a nef adelic line bundle $\overline{L}_f \in \widehat{\mathrm{Pic}}(X/k)_{\mathbf{Q}}$ such that $f^*\overline{\mathcal{L}}_f = q\overline{\mathcal{L}}_f$. The details can be found in [YZ24, Section 6.1].

This $\overline{L}_f$ will induce a dynamical canonical height on X (once we set up the height theory in the next section, Definition 3.1.1).

For the functoriality properties, we first suppose that $f : X' \to X$ is a morphism of essentially quasiprojective varieties over K. Then there is a *pullback functor/map* $f^* : \widehat{\mathrm{Pic}}(X/k) \to \widehat{\mathrm{Pic}}(X'/k)$ defined as follows. It suffices to define it in the case when $X = \mathcal{U}$ and $X' = \mathcal{U}'$ are quasiprojective over k. If $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, l_i))$ is a Cauchy sequence representing $\overline{\mathcal{L}} \in \widehat{\mathrm{Pic}}(\mathcal{U}/k)$, then for all i there is a projective model $\mathcal{X}_i'$ of $\mathcal{U}'$ along with a map $f_i : \mathcal{X}_i' \to \mathcal{X}$ of projective varieties over k, which is achieved by taking a rational map $\mathcal{X}_i' \dashrightarrow \mathcal{X}_i$ of projective models induced by $f : \mathcal{U}' \to \mathcal{U}$ and blowing up the indeterminacy locus. Then the pullback $f^*\overline{\mathcal{L}}$ is defined as the Cauchy sequence $(f^*\mathcal{L}, (\mathcal{X}_i', f_i^*\overline{\mathcal{L}}_i, f^*l_i))$ (of course one needs to prove that this is indeed a Cauchy sequence in the sense of the Definition 2.3.36. The pullback preserves the properties of being strongly nef, nef, and integrable.

Apart from pullback, we can also *vary the base.* This construction will be used in Lemma 3.1.14 and Theorem 3.1.15. We will sketch two cases. Details can be found at [YZ24, Section 2.5.5].

Example 2.3.42. First, suppose $k = \mathbf{Z}$. Let X be an quasiprojective variety over a number field K, and let $\mathcal{X}$ be a projective model of X over $\mathbf{Z}$. Then the generic fiber $\mathcal{X}_{\mathbf{Q}}$ is a projective model of $X_{\mathbf{Q}} = X$ over $\mathbf{Q}$. This induces a group homomorphism $\widehat{\mathrm{Pic}}(X/\mathbf{Z}) \to \widehat{\mathrm{Pic}}(X/\mathbf{Q}), \overline{L} \mapsto \widetilde{L}$ via the pullback morphism $\widehat{\mathrm{Pic}}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_{\mathbf{Q}})$ (notice that once we base change to over $\mathbf{Q}$, we lose any arithmetic information). It preserves the properties of being strongly nef, nef, and integrable. We call this morphism *taking the generic fiber.* There is a similar map for adelic divisors.

Second, let k be a field and K a finitely generated extension of k. Let X be an essentially quasiprojective variety over K. Then we want to define a map $\widehat{\mathrm{Pic}}(X/k) \to \widehat{\mathrm{Pic}}(X/K)$. This essentially uses Lemma 2.3.29. We can assume that X is isomorphic to the generic fiber of a morphism $\mathcal{U} \to \mathcal{V}$, where $\mathcal{U}$ is a quasiprojective model of X over k and $K(\mathcal{V}) = K$. We can extend this to a morphism of projective models $\mathcal{U}' \to \mathcal{V}'$ over k, and then $\widehat{\mathrm{Pic}}(X/k) \to \widehat{\mathrm{Pic}}(X/K)$ is induced by the pullback morphism $\mathrm{Pic}(\mathcal{U}') \to \mathrm{Pic}(\mathcal{U}'_\eta)$, where η is the generic point of $\mathcal{V}'$. Once again, there is a similar map for adelic divisors.

We can now sketch the intersection theory as in [YZ24, Chapter 4]. Just as one would expect from Definition 2.3.23 in the theory of Zhang's classical adelic line bundles on projective varieties, there is a top intersection number of adelic line bundles, and it is defined via a limit.

Definition 2.3.43. Let X/k be an essentially quasiprojective variety over k, and let d be the absolute dimension of a quasiprojective model of X over k. In particular, if $k = \mathbf{Z}$ and X is a quasiprojective variety over a number field, then $d = \dim(X) + 1$. Then there is a canonical symmetric and multilinear intersection product

$$\widehat{\mathrm{Pic}}(X/k)^d_{int} \to \mathbf{R}, \quad (\overline{L}_1, \ldots, \overline{L}_d) \mapsto \widehat{\deg}(\overline{L}_1 \cdots \overline{L}_d) = \overline{L}_1 \cdots \overline{L}_d$$

with the property such that if $\overline{L}_1, \ldots, \overline{L}_d$ are nef, then $\overline{L}_1 \cdots \overline{L}_d \geq 0$.

To define this, it suffices to take $X = \mathcal{U}$ a quasiprojective variety over k and take the $\overline{L}_i$ as strongly nef. Then expressing these adelic line bundles in the divisor language, i.e. Cauchy sequences of model adelic divisors on common projective models of $\mathcal{U}$, the intersection number is defined as the limit of the classical Arakelov-theoretic intersection numbers of those model adelic divisors. The details are worked out in [YZ24, Proposition 4.1.1].

Remark 2.3.44. In fact, one of the adelic line bundles in the intersection product does not need to be integrable, as long as the other $d-1$ are.

As one might expect from classical intersection theory, there is a projection formula for intersections of adelic line bundles, which is proved in the same way as always (prove it first in the model projective case, and then pass to the limit).

Theorem 2.3.45. Let k be either $\mathbf{Z}$ or a field. Let $f : X' \to X$ be a morphism of essentially quasiprojective varieties over k.[23] If the absolute dimensions of quasiprojective models of X' and X over k are all equal to d, and $\overline{L}_1, \ldots, \overline{L}_d \in \widehat{\mathrm{Pic}}(X/k)_{int}$, then

$$f^*\overline{L}_1 \cdots f^*\overline{L}_d = \begin{cases} \deg(f)(\overline{L}_1 \cdots \overline{L}_d) & f \text{ is dominant} \\ 0 & \text{otherwise} \end{cases}.$$

[23] There is a flatness assumption in [YZ24, Proposition 4.1.2], but note that in our special definition of essentially quasiprojective, flatness is automatic.

Finally, we will need the Deligne pairing of adelic line bundles, which again is a refinement of the intersection number as in Definition-Theorem 2.3.43.

Definition 2.3.46. Let X and Y be essentially projective (integral) varieties over k, and let $f : X \to Y$ be a projective flat morphism of relative dimension n. Assume that Y is normal. Then there is a symmetric multilinear functor, called the *Deligne pairing*,

$$\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{int}^{n+1} \to \widehat{\mathcal{P}\mathrm{ic}}(Y/k)_{int}, \quad (\overline{L}_1, \ldots, \overline{L}_{n+1}) \mapsto f_*\langle \overline{L}_1, \ldots, \overline{L}_{n+1}\rangle.$$

The Deligne pairing is compatible with base changes $Y' \to Y$ in the obvious manner, assuming that Y' is normal and $X_{Y'}$ is integral. Moreover, the Deligne pairing of strongly nef (resp. nef) line bundles is strongly nef (resp. nef).

This is [YZ24, Theorem 4.1.3], and its proof occupies the rest of Chapter 4 in that book. The general idea of the proof is, as usual, to approximate the line bundles with (Hermitian) line bundles on projective models of X and Y, apply the classical Deligne pairing as worked out in Section 2.2.4, and then show that these "model Deligne pairings" form a Cauchy sequence. Since the top intersection number is also defined in the same way, this implies that if the absolute dimension of a quasiprojective model of Y over k is 1, so that the absolute dimension of a quasiprojective model of Y over k is $n + 1$, then $\widehat{\deg}(f_*\langle \overline{L}_1, \ldots, \overline{L}_{n+1}\rangle) = \overline{L}_1 \cdots \overline{L}_d$.

There are also many other functoriality properties of the Deligne pairing, e.g. a projection formula in various settings. Since we only occasionally need certain special cases in these notes, we omit them here and instead refer to [YZ24, Theorem 4.6.1].

Remark 2.3.47. It should be noted that the Deligne pairing as defined immediately above is compatible with the Deligne pairing from Section 2.2.4 for Hermitian line bundles and the Deligne pairing as mentioned in Section 2.3.2 for Zhang's classical adelic line bundles on projective varieties. Of course, we have not yet explained how Zhang's classical adelic line bundles on projective varieties are adelic line bundles in the sense of this section; that will be explained in the next Section 2.3.4.

2.3.4 Analytification and admissible metrics revisited

At this point it may be rather unclear how the p-adic metrics that we needed for Zhang's classical adelic line bundles on projective varieties, from Section 2.3.2, appear in the "new" adelic line bundle theory as explained in Section 2.3.3. This is explained by *analytification* of adelic line bundles using *Berkovich analytic spaces*. The intuition, as vaguely indicated in [Yua24, Appendix A.5], is that the Berkovich space "includes the data" of all metrized reduction graphs of all regular projective models of an essentially quasiprojective variety over k, in that there are natural maps from such reduction graphs to an appropriate Berkovich analytification. Therefore we will be able to transfer all of the p-adic data onto the Berkovich space.

The main reference for this section is [YZ24, Chapter 3] along with [Yua24, Appendix A].

We quickly review the definition of Berkovich space. Let k be a commutative Banach ring with norm $|\cdot|_{Ban}$; that is, k is complete under the metric induced by $|\cdot|_{Ban}$. Let X be a k-scheme. Then

Definition 2.3.48. The *Berkovich analytification* X^{an} of X is constructed as follows. If $X = \mathrm{Spec}(A)$ is affine, then X^{an} as a set is the space $\mathcal{M}(A/k)$ of multiplicative seminorms on A whose restriction to k is bounded by $|\cdot|_{Ban}$. For $x \in \mathcal{M}(A/k)$, we denote the seminorm corresponding to x by $|\cdot|_x$. The topology on $\mathcal{M}(A/k)$ is the weakest one such that for all $f \in A$, the map $\mathcal{M}(A/k) \to \mathbf{R}$, $x \mapsto |f|_x$ is continuous.

In general, we cover X by affine open subschemes $\mathrm{Spec}(A_i)$, and we construct X^{an} by gluing the $\mathcal{M}(A_i/k)$ in the canonical way. The topology on X^{an} is the weakest one such that each $\mathcal{M}(A_i/k)$ is open.

In our applications, we will always either take $k = \mathbf{Z}$ with the Euclidean norm, or k to be a field with the trivial norm, and we will equip k with these norms without further mention. Examples of the Berkovich analytification in various basic cases can be found at [YZ24, Section 3.1.2].

Remark 2.3.49. A basic remark is that as sets, we can decompose X^{an} into a disjoint union $X^{an}[\infty] \cup X^{an}[f]$, where $X^{an}[\infty]$ is the subset of Archimedean seminorms, and $X^{an}[f]$ is the subset of non-Archimedean seminorms. For example, if k is a field equipped with the trivial norm, then $X^{an}[\infty]$ is empty since a (semi)norm is non-Archimedean if it is bounded by 1 on multiples of the multiplicative identity 1. We also note that the Berkovich analytification is functorial: a map $f : X \to Y$ of k-schemes naturally induces a continuous map $f^{an} : X^{an} \to Y^{an}$.

We would also like to attach some ring-theoretic objects to X^{an}, although it is not a ringed space. We start in the affine case $X = \mathrm{Spec}(A)$. For $x \in X^{an}$, the kernel of $|\cdot|_x$ in A is a prime ideal by multiplicativity. Then this observation gives a *contraction map* $X^{an} \to X$, whose local definition on affines is that the image $\overline{x}$ of $x \in \mathcal{M}(A/k) \subseteq X^{an}$ is precisely the prime ideal $|\cdot|_x$ of $\mathrm{Spec}(A)$.

Continuing the above observation, we also define the *residue field* H_x at x to be the completion of $\mathrm{Frac}(A/\ker(|\cdot|_x))$ with respect to the norm induced by $|\cdot|_x$. Since in general, X^{an} is constructed from gluing various $\mathcal{M}(A_i/k)$, this definition can be extended to points of general X^{an} as well. Hence if $f : X \to Y$ is a map of k-schemes, then we can define the *fiber* of the map $f^{an} : X^{an} \to Y^{an}$ of Berkovich analytic spaces as follows: for $y \in Y^{an}$, the topological fiber $(f^{an})^{-1}(y)$ is homeomorphic to $(X_{H_y}/H_y)^{an}$, where X_{H_y} is the base change of X with respect to the natural map $\mathrm{Spec}(H_y) \to Y$.

We need to define one more map, the *reduction map* $r : X^{an} \to X$. It will be used in the definition of the analyfication of adelic divisors.

Definition 2.3.50. Suppose X is *proper* over k. We will define the *reduction map* $r : X^{an} \to X$ as follows, which will be a continuous map of topological spaces. First, suppose $x \in X^{an}[f]$, so that H_x is a complete non-Archimedean field with valuation ring R_x. By the valuative criterion of properness, the natural map $\text{Spec}(H_x) \to X$ extends to a map $\text{Spec}(R_x) \to X$. Then the image of the closed point of $\text{Spec}(R_x)$ is defined to be $r(x)$.

Otherwise, suppose $x \in X^{an}[\infty]$, so that H_x is either $\mathbf{R}$ or $\mathbf{C}$. Then the image of $\text{Spec}(H_x) \to X$ is defined to be $r(x)$.

We will now discuss arithmetic divisors and metrized line bundles on Berkovich spaces. This will closely resemble the classical Arakelov theory from Section 2.2. Let k be a commutative Banach ring and X an integral k-variety.

Definition 2.3.51. Let D be a Cartier divisor on X. We define a *Green's function* for D on X^{an} as a continuous function $g : X^{an} - |D|^{an} \to \mathbf{R}$ such that for any local equation f cutting out D on an open subscheme $U \subseteq X$, $g + \log|f|$ extends to a continuous function on U^{an}. Here $\log|f|$ is the function on U^{an} that takes in a seminorm $|\cdot|_x$ and outputs $\log|f|_x$.

A pair $\overline{D} = (D, g)$ where g is a Green's function for the Cartier divisor X is called an *arithmetic divisor*. It is called *effective* if D is effective on X and g is nonnegative. We then have a group $\widehat{\text{Div}}(X^{an})$ of arithmetic divisors on X^{an}, as well as a subgroup $\widehat{\text{Pr}}(X^{an})$ consisting of principal arithmetic divisors of the form $(\text{div}(f), -\log|f|)$ for a nonzero rational function f on X. We call the quotient $\widehat{\text{CaCl}}(X^{an})$.

We do not want to consider all possible arithmetic divisors on X^{an}. In particular, points of the Berkovich space X^{an} often come in "legs/intervals," where if $|\cdot|$ is a multiplicative seminorm on an affine subscheme of X whose restriction to k is bounded by $|\cdot|_{Ban}$, then the same is true for $|\cdot|^t$ where t is any nonnegative real number (or perhaps only if $t \in [0, 1]$). Therefore we would like our arithmetic divisors to have reasonable interactions with these types of points, which motivates:

Definition 2.3.52. An arithmetic divisor $\overline{D} = (D, g)$ is called *norm-equivariant* if $x, y \in X^{an} - |D|^{an}$ affine-locally satisfy $\|\cdot\|_x = \|\cdot\|_y^t$ for some $t \geq 0$, then $g(x) = tg(y)$. Norm-equivariant arithmetic divisors form a subgroup $\widehat{\text{Div}}(X^{an})_{eqv}$ of $\widehat{\text{Div}}(X^{an})$.

Note that principal arithmetic divisors are norm-equivariant by construction, so we can form a subquotient $\widehat{\text{CaCl}}(X^{an})_{eqv}$ in the evident way.

We now turn to the metrized line bundle side of things. Most of the constructions are very similar again.

Definition 2.3.53. With the above notation, let L be a line bundle on X. Recall for $x \in X^{an}$, $\overline{x}$ means the image of x under the contraction map $X^{an} \to X$. We define the *fiber* $L^{an}(x)$ of L at x to be the 1-dimension H_x-vector space $L(\overline{x}) \otimes_{k(\overline{x})} H_x$. Note that $k(\overline{x})$ is simply the fraction field of the domain $A/\ker(|\cdot|_x)$, $\mathrm{Spec}(A)^{an}$ being an open neighborhood of x in X^{an}. We define a *metric* $\|\cdot\|$ of L on X^{an} to be a collection of H_x-metrics[24] on the fibers $L^{an}(x)$ that is also continuous in the sense that for any rational section l on $U \subseteq X$, the function $x \mapsto \|l(x)\|_x$ is continuous on U^{an}.

A pair $\overline{L} = (L, \|\cdot\|)$ as above is called a *metrized line bundle* on X^{an}. We then have a group $\widehat{\mathrm{Pic}}(X^{an})$ of metrized line bundles.

It is not hard to see that there is a natural isomorphism $\widehat{\mathrm{CaCl}}(X^{an}) \cong \widehat{\mathrm{Pic}}(X^{an})$. As is familiar by now, the map from left to right is given by $(D, g) \mapsto (\mathcal{O}(D), \exp(-g))$, and the reverse map is given by $(L, \|\cdot\|) \mapsto (\mathrm{div}(s), -\log\|s\|)$ for any rational section s of L. This then allows us to transfer the norm-equivariance to metrized line bundles.

Definition 2.3.54. An metrized line bundle $\overline{L} = (L, \|\cdot\|)$ is called *norm-equivariant* if for all rational sections s of L and all $x, y \in X^{an} - |\mathrm{div}(s)|^{an}$ affine-locally satisfying $\|\cdot\|_x = \|\cdot\|_y^t$ for some $t \geq 0$, we have $\|s(x)\|_x = \|s(y)\|_y^t$. Norm-equivariant metrized line bundles form a subgroup $\widehat{\mathrm{Pic}}(X^{an})_{eqv}$ of $\widehat{\mathrm{Pic}}(X^{an})$, which is isomorphic to $\widehat{\mathrm{CaCl}}(X^{an})_{eqv}$.

Remark 2.3.55. By the norm-equivariance condition, Zhang's classical adelic line bundles on projective varieties X over number fields may be thought of as metrized line bundles in the sense of Definition 2.3.53, by taking the analytification of $X/\mathbf{Z}$ and extending the metrics via norm-equivariance.

Now we discuss how to transfer the adelic line bundles of the previous section to this more classical formulation with Green's functions and metrics. The key theorem is the construction of the *analytification* maps, which is done in [YZ24, Sections 3.3–3.4].

Theorem 2.3.56. Let k be $\mathbf{Z}$ or a field, and X an essentially quasiprojective variety over k. Then there are canonical *injective* analytification maps

$$\widehat{\mathrm{Div}}(X/k) \to \widehat{\mathrm{Div}}(X^{an})_{eqv}, \quad \widehat{\mathrm{CaCl}}(X/k) \to \widehat{\mathrm{CaCl}}(X^{an})_{eqv}, \quad \widehat{\mathrm{Pic}}(X/k) \to \widehat{\mathrm{Pic}}(X^{an})_{eqv}.$$

We will use this theorem to great effect in Section 3.2.2, where we will use the fact that to specify an adelic divisor or adelic line bundle on X, it suffices to describe its analytification on X^{an} by the injectivity.

The theorem is proved in the following way. Take the statement for divisors as an example. We start in the case when $X = \mathcal{X}$ is projective over k, and we define the map single (model) adelic divisors $\overline{D} \in \widehat{\mathrm{Div}}(\mathcal{X})$. The underlying line bundle of the analyfication

[24] This means that for all $f \in H_x$ and $l \in L^{an}(x)$, $\|fl\|_x = |f|_x \, \|l\|_x$.

will still be D. For the analytified Green's function $\widetilde{g}$, we define it separately on $x \in X^{an}[f]$ and $X^{an}[\infty]$. For $x \in X^{an}[f]$, by the properness of $\mathcal{X}$ over k, we may define the reduction $r(x) \in X$. Pick an open neighborhood $\mathcal{U}$ of $r(x)$ such that D is cut out by a single equation $f \in K(\mathcal{U})^\times$ on U. Since $r(x) \in \mathcal{U}$, it follows that $x \in U^{an}$, and we define $\widetilde{g}(x) := -\log|f(x)|$, which turns out to be independent of all the choices made. Finally, for $x \in X^{an}[\infty]$, which only applies if $k = \mathbf{Z}$ and $\overline{D} = (D, g)$ is an arithmetic divisor, the definition of $\widetilde{g}(x)$ is induced from g by requiring norm-equivariance. It turns out that $\widetilde{g}$ is continuous on X^{an} and is a Green's function for D.

This finishes the construction for projective models. If $X = \mathcal{U}$ is now quasiprojective over k and $\overline{D} \in \widehat{\mathrm{Div}}(\mathcal{U}/k)$, we apply the above construction to a Cauchy sequence of model adelic divisors, and the resulting arithmetic divisors on $\mathcal{U}^{an}$ will converge (namely, the Green's functions will converge) by the Cauchy condition. Finally, the essentially quasiprojective case follows from the quasiprojective case by taking representatives in the direct limit.

To finish this deluge of preliminary definitions and results, we need to discuss [Yua24, Theorem A.1] that describes admissible metrized line bundles in our new setting. Recall from Section 2.3.1 that admissibility of a metrized line bundle was originally described by some curvature condition, either on Riemann surfaces or reduction graphs. The language of Berkovich spaces no longer uses these constructions, but it is still possible to define admissible metrics in the cases that we care about. We summarize in a theorem:

Theorem 2.3.57. Let C be a smooth projective curve of genus $g > 0$ over a non-Archimedean field K. Then there are unique *admissible* metrics $\|\cdot\|_a$ and $\|\cdot\|_{\Delta,a}$ of $\omega_{C/K}$ and $\mathcal{O}_{C^2}(\Delta)$ on C^{an} and $(C^2)^{an}$ respectively, which are induced from a canonical retraction map $C^{an} \to \Gamma(C)$ (see the discussion in [Yua24, Appendix A.5]). Here $\Gamma(C)$ is the reduction graph of C.

In other words, the admissible metrics from [Zha93] uniquely determine the $\|\cdot\|_a$ and $\|\cdot\|_{\Delta,a}$ in the above theorem.

The above theorem is a local result, but we would like a *relative* version for families of curves. This is achieved by [Yua24, Theorem 2.3], which defines two of the key adelic line bundles we will need in the next section.

Theorem 2.3.58. Let k be $\mathbf{Z}$ or a field, and let S be a quasiprojective flat normal integral scheme over k. Let $\pi : X \to S$ be a smooth relative curve (a projective flat morphism of relative dimension 1 with geometrically connected fibers). Suppose the fibers have genus $g > 0$, and let $\Delta : X \to X \times_S X$ be the diagonal morphism. Then:

(1) There is an adelic line bundle $\overline{\omega}_{X/S,a} \in \widehat{\mathcal{P}\mathrm{ic}}(X/k)$ with underlying line bundle $\omega_{X/S}$, such that for any $v \in S^{an}$, the metric of the base change $\omega_{X_{H_v}/H_v}$ on the fiber $X_{H_v}^{an}$ induced by the analytification of $\omega_{X/S,a}$ is equal to the canonical admissible metric $\|\cdot\|_a$ as described in Theorem 2.3.57. Moreover, $\overline{\omega}_{X/S,a}$ is nef and unique up to isomorphism.

(2) There is an adelic line bundle $\overline{\mathcal{O}}(\Delta)_a \in \widehat{\mathcal{P}\mathrm{ic}}(X \times_S X/k)$ with underlying line bundle $\mathcal{O}(\Delta)$, such that for any $v \in S^{an}$, the metric of the base change $\mathcal{O}(\Delta_{H_v})$ on the fiber $(X^2_{H_v})^{an}$ induced by the analytification of $\overline{\mathcal{O}}(\Delta)_a$ is equal to the canonical admissible metric $\|\cdot\|_{\Delta,a}$ as described in Theorem 2.3.57. Moreover, $\overline{\mathcal{O}}(\Delta)_a$ is integrable and unique up to isomorphism.

(3) These definitions are compatible in the sense that the canonical isomorphism $\omega_{X/S} \xrightarrow{\sim} \Delta^*\mathcal{O}(-\Delta)$ induces an isomorphism of adelic line bundles $\overline{\omega}_{X/S,a} \xrightarrow{\sim} \Delta^*\overline{\mathcal{O}}(-\Delta)_a$, where $\overline{\mathcal{O}}(-\Delta)_a$ is the inverse of $\overline{\mathcal{O}}(\Delta)_a$.

3 Bigness and heights

3.1 Preliminaries and the height inequality

In this section k will be either $\mathbf{Z}$ or a field, and also we let K be a number field or function field of one variable over k, respectively.

Our first goal is to define the height of a point on a quasiprojective variety X/K with respect to an adelic line bundle $\overline{L} \in \widehat{\mathrm{Pic}}(X/k)$, and to give some tools for analyzing it. This will be a generalization of the discussion from Section 2.3.3, and indeed the basic definitions will formally look the same. We will follow Chapter 5 of [YZ24].

Definition 3.1.1. Let X be a quasiprojective variety over K and let $x \in X(\overline{K})$. Let $\overline{L}$ be an integrable adelic line bundle, i.e. $\overline{L} \in \widehat{\mathrm{Pic}}(X/k)_{int}$. We define the *height* of x by

$$h_{\overline{L}}(x) := \frac{\widehat{\deg}\langle \overline{L}|_x \rangle}{[K(x) : K]}.$$

Technically, we take the Deligne pairing of relative dimension 0 from $\widehat{\mathrm{Pic}}(x/k)$ (with $x = \mathrm{Spec}(K(x))$ considered as a closed point in X) to $\widehat{\mathrm{Pic}}(K/k)$, and then the Arakelov degree function (i.e. the 0-dimensional top intersection number). This is solely because by removing the application of $\widehat{\deg}$, we can define a "vector-valued height" whose output is an object in $\widehat{\mathrm{Pic}}(K/k)$, which is valid in the more general case when K is only a finitely generated field over $\mathrm{Frac}(k)$ (i.e. we allow transcendence degree > 1). On the other hand, for our purposes we will only need the height "as a number."

Remark 3.1.2. The definition can easily be extended to integrable $\mathbf{Q}$-adelic line bundles. Also, in the future, whenever we introduce a height function associated to an adelic line bundle $\overline{L}$, we will implicitly assume that it is integrable without further mention. The former will not cause any problems as integrable line bundles comprise a very large class of adelic line bundles, and all of the operations we might do to them (e.g. pullback, base change, Deligne pairing) preserve this property.

Remark 3.1.3. Note that this is not normalized like the previous heights; it depends on the field K. In particular, when K is a number field, it is off by a factor of $[K : \mathbf{Q}]$ from the previously defined heights, e.g. Definition 2.3.24. Also, as in the projective case there is an easy extension of this to height of higher-dimensional closed subvarieties of X. On the other hand, in the quasiprojective case we must stipulate that those subvarieties are projective, since that is no longer automatic.

Remark 3.1.4. As we would expect, we have the following functoriality property for heights. Suppose $\pi : X \to S$ is a morphism of quasiprojective varieties over K and $\overline{M}$ an (integrable) adelic line bundle on S. Then for $x \in X(\overline{K})$, $\pi(x) \in S(\overline{K})$, and

$$h_{\pi^*\overline{M}}(x) = \frac{\widehat{\deg}\langle(\pi^*\overline{M})|_x\rangle}{[K(x) : K]} = \frac{\widehat{\deg}\langle\overline{M}|_{\pi(x)}\rangle[K(x) : K(\pi(x))]}{[K(x) : K]} = h_{\overline{M}}(\pi(x))$$

via the projection formula Theorem 2.3.45.

Let's unravel Definition 3.1.1. The numerator can be understood better in terms of metrized line bundles. Consider the analytification $\overline{L}^{an} = (L, \|\cdot\|)$ of $\overline{L}$. We have

$$h_{\overline{L}}(x) = -\frac{1}{[K(x) : K]} \sum_{v \in M_K} \sum_{z \in O(x)_{\overline{K}_v}} \log\left(\|s(z)\|^{\deg_{K_v}(z)}\right). \tag{3.1.1}$$

for any rational section s of L without a zero or pole at x. We need to clarify some notations. If K is a number field, then M_K has the usual meaning of the set of places of K, but if K is a function field of one variable over k, M_K means the set of places of K *that are trivial on* k. Next, $O(x)_{\overline{K}_v}$, for a place $v \in M_K$, means the image of the Galois orbit $\mathrm{Gal}(\overline{K}/K)\cdot x \subseteq X(\overline{K})$ under the natural map to $X(\overline{K}_v)$, where we have fixed an embedding $K_v \hookrightarrow \overline{K}_v$. Note that a point $z \in O(x)_{\overline{K}_v}$ is naturally a point in the fiber of X^{an} over $v \in \mathrm{Spec}(K)^{an}$, so $\|s(z)\|$ makes sense.

Proof of Equation 3.1.1. The proof of this formula essentially follows from unraveling the definitions. It is very similar to the computation in Example 2.3.25. We will sketch it in the case $k = \mathbf{Z}$, but the case where k is a field is the exact same.

To start, consider any adelic line bundle $\overline{L} \in \widehat{\mathrm{Pic}}(K/k)$. We would like to give a formula for $\widehat{\deg}(\overline{L})$ in terms of its analytification. If $(\mathrm{Spec}(\mathcal{O}_K), \overline{\mathcal{L}})$ is a Hermitian line bundle approximating $\overline{L}$ in a representing Cauchy sequence, then $\widehat{\deg}(\overline{\mathcal{L}})$ is simply the usual Arakelov degree $\sum_{\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_K)} \mathrm{ord}_{\mathfrak{p}}(s) \log(n_{\mathfrak{p}}) - \sum_{\sigma \in K(\mathbf{C})} \log \|s\|_\sigma$ for a rational section s of $\mathcal{L}$. But by the analytification procedure and comparison between arithmetic divisors and metrized line bundles on Berkovich space as explained in Section 2.3.4, this is

$$\widehat{\deg}(\overline{\mathcal{L}}) = -\sum_{v \in M_K} \log\left(\|s(v)\|^{e_v}\right)$$

where $v \in M_K$ is interpreted as a point in $\mathrm{Spec}(K)^{an}$ by taking it to be the normalized v-adic valuation of K, and $\|\cdot(v)\|$ is the metric of $\mathcal{L}^{an}$ at v.[25] By passing to the limit, the same formula holds for $\overline{L}$ in place of $\overline{\mathcal{L}}$.

We go back to the setting where $\overline{L}$ is an adelic line bundle in $\widehat{\mathrm{Pic}}(X/k)$ with a Hermitian line bundle $\overline{\mathcal{L}}$ on a projective model $\mathcal{X}/\mathcal{O}_K$ of (a quasiprojective model of) X approximating $\overline{L}$. We need to describe the analytification of $\overline{\mathcal{L}}|_x$ in terms of the analytification of $\overline{\mathcal{L}}$. Let s be a rational section of $\mathcal{L}$ without a zero or pole at x, and we consider the pullback of $\mathcal{L}$ and s to $\mathrm{Spec}(\mathcal{O}_{K(x)})$. Then if v is a finite place of K, then the v-adic contribution to $\widehat{\deg}\langle\widehat{\mathrm{div}}(s|_{\mathrm{Spec}(\mathcal{O}_{K(x)})})\rangle$ (where $\langle\cdot\rangle$ denotes the Deligne pairing of relative dimension 0 from $\mathrm{Spec}(\mathcal{O}_{K(x)})$ to $\mathrm{Spec}(\mathcal{O}_K)$, i.e. the norm functor) is given by

$$\sum_{w|v} \mathrm{ord}_w(s|_{\mathrm{Spec}(\mathcal{O}_{K(x)})})[\kappa(w):\kappa(v)]\log(n_v) = \sum_{w|v} \mathrm{ord}_w(s|_{\mathrm{Spec}(\mathcal{O}_{K(x)})})\log(n_w) = -\sum_{w|v}\log\|s(w)\|$$

with $\kappa(w), \kappa(v)$ the residue fields at w and v and $\|\cdot(w)\|$ is the metric of $\mathcal{L}|^{an}_{\mathrm{Spec}(\mathcal{O}_{K(x)})}$ at w. Using the canonical bijection between $O(x)_{\overline{K}_v}$ and the set of $w \in M_{K(x)}$ extending v, we get the desired equality

$$\sum_{w|v}\log\|s(w)\| = \sum_{z\in O(x)_{\overline{K}_v}} \log\left(\|s(z)\|^{\deg_{K_v}(z)}\right).$$

The exponent comes from the fact that if w corresponds to $z \in O(x)_{\overline{K}_v}$, then the normalized w-adic absolute value on the left-hand side is exactly the $\deg_{K_v}(z)$ power of the v-adic absolute value (affine-locally on X) induced from $z : \mathrm{Spec}(\overline{K}_v) \to X$.[26]

Finally, if v is an infinite place of K there is a similar argument for the above equality by using the Hermitian metrics of $\mathcal{L}$, and once again by passing to the limit, we have the proof for $\overline{L}$ in place of $\overline{\mathcal{L}}$. □

In light of this last formula 3.1.1, we see that it is important to get upper bounds on $\|s(x)\|$ for global sections s of L, which translate to lower bounds on $h_{\mathcal{L}}(x)$ away from the vanishing locus on s. In particular if $s \in H^0(X, L)$ with $\sup_{x\in X^{an}}\|s(x)\| \le 1$, then $h_{\mathcal{L}}(x)$ is nonnegative on a dense open subvariety of X, away from the support of $\mathrm{div}(s)$. This motivates the following definition, which should be very reminiscent of Definition 2.2.4:

Definition 3.1.5. Let X be an essentially quasiprojective variety over k. Let $\overline{L}$ be an adelic line bundle on X, and by abuse of notation also denote by $\overline{L}$ its analytification on X^{an}. We define

$$\widehat{H}^0(X, \overline{L}) := \{s \in H^0(X, L) : \sup_{x\in X^{an}}\|s(x)\| \le 1\}.$$

[25] See Section 2.1 for the normalizations.

[26] We note that $\overline{K}_v$ carries the literal extension of the natural v-adic absolute value on K_v, so the restricted absolute values on the intermediate finite extensions of K_v inside $\overline{K}_v$ are not normalized, in our sense. This explains the exponent term.

We call elements of $\widehat{H}^0(X, \overline{L})$ *effective* or *small*. We also define

$$\widehat{h}^0(X, \overline{L}) \coloneqq \log \# \widehat{H}^0(X, \overline{L}).$$

If $\widehat{h}^0(X, \overline{L}) > 0$ then we call $\overline{L}$ *effective*.

It can be shown ([YZ24, Lemma 5.1.2–Definition 5.1.3]) that $\widehat{H}^0(X, \overline{L}) = \{s \in H^0(X, L) : \widehat{\mathrm{div}}(s)$ is effective$\}$, where the effectivity can be checked on a quasiprojective representative of $\overline{L}$. As in the classical case, $\widehat{h}^0(X, \overline{L})$ is always finite via essentially the same proof.

We can certainly rewrite this definition in terms of adelic divisors instead, which has the advantage of being easier to use in practice. We essentially already gave it in Definition 2.3.31.

Definition 3.1.6. Let X be an essentially quasiprojective variety over k, and $\overline{D} \in \widehat{\mathrm{Div}}(X/k)$. Suppose that $\overline{D}$ is represented by an adelic divisor in $\widehat{\mathrm{Div}}(\mathcal{U}/k)$ (by abuse of notation, also denoted by $\overline{D}$) for some quasiprojective model $\mathcal{U}$ of X. Then we say $\overline{D}$ is effective if it can be written as a Cauchy sequence of model adelic divisors, all of which are effective on their projective models.

It is not difficult to see that the effectivity of $\overline{D}$ is equivalent to the effectivity of $\mathcal{O}(\overline{D})$. Moreover, with this setup in hand, we can record an important criterion for effectivity:

Lemma 3.1.7. [YZ24, Lemma 5.1.2] Let X be an essentially quasiprojective variety over k, and $\overline{D} \in \widehat{\mathrm{Div}}(X/k)$. Then $\overline{D}$ is effective if and only if its analytification $\overline{D}^{an} \in \widehat{\mathrm{Div}}(X^{an})$ is effective. If X is normal, this is true if and only if the Green's function $g_{\overline{D}}$ of $\overline{D}^{an}$ is nonnegative on $X^{an} - \mathrm{supp}(D)^{an}$.

The existence of small sections is the fundamental property we would like an adelic line bundle $\overline{L}$ to have, so that its associated height function has at least the basic properties one might want from a height function. On the other hand, we see from the Definition 3.1.1 that $mh_{\overline{L}} = h_{m\overline{L}}$, and so it is enough to show that some multiple of $\overline{L}$ has a small section. This "asymptotic" existence turns out to be equally as difficult as showing that $\overline{L}$ has a section (if it has one at all). Therefore we have the two key definitions:

Definition 3.1.8. [YZ24, Theorem 5.2.1] Let X be an essentially quasiprojective variety over k, such that d is the *absolute* dimension of a quasiprojective model of X over k. If $\overline{L}$ is an adelic line bundle on X, we define

$$\widehat{\mathrm{vol}}(X, \overline{L}) \coloneqq \lim_{m \to \infty} \frac{d!}{m^d} \widehat{h}^0(X, m\overline{L}).$$

In particular, this limit exists, and it can be computed via model adelic line bundles: if $\overline{L}$ is given by a Cauchy sequence $(\mathcal{L}, \mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_i$, then

$$\widehat{\mathrm{vol}}(X, \overline{L}) = \lim_{i \to \infty} \widehat{\mathrm{vol}}(\mathcal{X}_i, \overline{\mathcal{L}}_i),$$

where the volumes on the right-hand side are (classical) volumes of $\mathbf{Q}$-hermitian line bundles in the arithmetic case $k = \mathbf{Z}$, or volumes of $\mathbf{Q}$-line bundles in the geometric case $k =$ field.

Definition 3.1.9. We say an adelic line bundle $\overline{L}$ is *big* if $\widehat{\mathrm{vol}}(X, \overline{L}) > 0$.

A good reference for bigness and volumes in the classical geometric setting is [Laz04, Section 2.2], or a quick summary can be found at [YZ24, Sections 5.1.3, 5.2.3]. Once again, is clear that if an adelic line bundle $\overline{L}$ is big, then $h_{\overline{L}}$ is nonnegative on a dense open subvariety of X. Therefore this arithmetic property of heights has been transferred to a geometric property.

We can now introduce the basic tools to discuss bigness of line bundles. The limit property of Definition-Theorem 3.1.8 allows us to deduce facts about volumes of adelic line bundles from corresponding facts about Hermitian line bundles. In the latter case, the following fundamental fact (stated slightly differently) is proved by Yuan in [Yua08, Theorem 2.2].

Theorem 3.1.10 (Arithmetic Siu inequality, Hermitian case)**.** Suppose $\mathcal{X}$ is an arithmetic variety of absolute dimension d, and $\overline{\mathcal{L}}, \overline{\mathcal{M}}$ are nef Hermitian line bundles on $\mathcal{X}$. Then for $n \in \mathbf{N}$,

$$\widehat{h}^0(n(\overline{\mathcal{L}} - \overline{\mathcal{M}})) \geq \frac{n^d}{d!}(\overline{\mathcal{L}}^d - d\overline{\mathcal{L}}^{d-1} \cdot \overline{\mathcal{M}}) + o(n^d).$$

In particular

$$\widehat{\mathrm{vol}}(X, \overline{\mathcal{L}} - \overline{\mathcal{M}}) \geq \overline{\mathcal{L}}^d - d\overline{\mathcal{L}}^{d-1} \cdot \overline{\mathcal{M}}.$$

Remark 3.1.11. A recent Diophantine application due to Yuan [Yua25] is a modified proof of the Mordell conjecture, following Vojta's original proof, using the arithmetic Siu inequality to replace the difficult application of Gillet–Soulé's arithmetic Riemann–Roch theorem.

We also have the arithmetic Hilbert–Samuel formula, which has many formulations due to many mathematicians (including Abbes–Bouche, Gillet–Soulé, Moriwaki–Zhang, Yuan). The one we will take can be found in [Mor14, Theorem 7.10].

Theorem 3.1.12 (Arithmetic Hilbert–Samuel formula, Hermitian case)**.** Suppose $\mathcal{X}$ is an arithmetic variety of absolute dimension d, and $\overline{\mathcal{L}}, \overline{\mathcal{M}}$ are Hermitian line bundles on $\mathcal{X}$ with $\overline{\mathcal{L}}$ nef. Then

$$\widehat{h}^0(\mathcal{X}, n\overline{\mathcal{L}} + \overline{\mathcal{M}}) = \frac{n^d}{d!}\overline{\mathcal{L}}^d + o(n^d).$$

In particular

$$\widehat{\mathrm{vol}}(X, \overline{\mathcal{L}}) = \overline{\mathcal{L}}^d.$$

Both Theorems 3.1.10 and 3.1.12 can be immediately extended to the adelic case by taking limits. We omit the statement since it is formally the same, except that X is now a flat quasiprojective integral scheme over k, and $\overline{L}$ is an adelic line bundle over X. See [YZ24, Theorem 5.2.2].

We need two more lemmas. The first is on the continuity of volume. Unfortunately the proof is also rather complicated, relying on the Fujita approximation theorem and the log-concavity of the volume function.

Theorem 3.1.13. Let $\overline{L}$, $\overline{M}_1, \ldots, \overline{M}_r$ be adelic $\mathbf{Q}$-line bundles on X. Then

$$\lim_{t_1,\ldots,t_r \to 0} \widehat{\mathrm{vol}}(\overline{L} + t_1\overline{M}_1 + \ldots + t_r\overline{M}_r) = \widehat{\mathrm{vol}}(\overline{L}),$$

where $t_1, \ldots, t_r$ are rational numbers converging to 0.

The second lemma concerns bigness of the generic fiber. It says that the bigness of the (geometric) "generic fiber" (whose construction we will recall below) of an adelic line bundle is close to the bigness of the adelic line bundle itself.

Lemma 3.1.14. Let K be a number field (resp. function field of one variable over k) if k is $\mathbf{Z}$ (resp. a field). Let $f : X \to K$ be a quasiprojective variety over K. Let $\overline{L} \in \widehat{\mathrm{Pic}}(X/k)$, write $\widetilde{L}$ for the image of $\overline{L}$ in $\widehat{\mathrm{Pic}}(X/K)$,[27] and let $\overline{N} \in \widehat{\mathrm{Pic}}(K/k)$ be an adelic line bundle with $\widehat{\deg}(\overline{N}) > 0$. Assume that $\widetilde{L}$ is big on X over K. Then for all sufficiently large rational numbers c, the adelic ($\mathbf{Q}$-)line bundle $\overline{L} + cf^*\overline{N} \in \widehat{\mathrm{Pic}}(X/k)$ is big.

The intuition behind this lemma is that the existence of small global sections of (a multiple) of $\widetilde{L}$ implies that the height function associated to $\widetilde{L}$ is lower bounded. One can see this by simply observing that $h_{\widetilde{L}}$ is a Weil height associated to the underlying line bundle L, which has a section and hence has lower bounded Weil height as in Proposition 2.1.3. So in the above notation, if c is large enough, then we might expect that the height associated to $\overline{L} + c\pi^*\overline{N}$ actually becomes nonnegative, which should result from its bigness.

We are now prepared to prove the main result of the section, which is the following *height inequality* [YZ24, Theorem 5.3.7]. This is the key input to our main theorems that relies on the framework of adelic line bundles. It was already stated as Theorem 1.2.1, but we re-copy it below:

Theorem 3.1.15. Let K be a number field (resp. function field of one variable over k) if k is $\mathbf{Z}$ (resp. a field). Let $\pi : X \to S$ be a morphism of quasiprojective varieties over K. Let $\overline{L} \in \widehat{\mathrm{Pic}}(X/k)$ and $\overline{M} \in \widehat{\mathrm{Pic}}(S/k)$ be adelic line bundles, and write $\widetilde{L}$ for the image of $\overline{L}$ in $\widehat{\mathrm{Pic}}(X/K)$.

[27] Concretely, one takes the generic fiber of $\overline{L}$ over $\mathbf{Z}$ in the arithmetic case (nothing to do in the geometric case), becoming an adelic line bundle in $\widehat{\mathrm{Pic}}(X_{\mathrm{Frac}(k)}/\mathrm{Frac}(k))$, and then extends the base to K. This is described in more detail in [YZ24, Section 2.5.5], where it is called "varying the base". Also see the description in Example 2.3.42.

(1) If $\overline{L}$ is big on X/k, then there exist $\epsilon > 0$ and a nonempty open subvariety U of X such that $h_{\overline{L}}(x) \geq \epsilon h_{\overline{M}}(\pi(x))$ for all $x \in U(\overline{K})$ (notice that there are no assumptions on $\overline{M}$ here!).

(2) If $\overline{L}$ is nef on X/k, and $\widetilde{L}$ is big on X/K, then for any $c > 0$ there exist $\epsilon > 0$ and a nonempty open subvariety U of X such that $h_{\overline{L}}(x) \geq \epsilon h_{\overline{M}}(\pi(x)) - c$ for all $x \in U(\overline{K})$.

(3) If $\widetilde{L}$ is big on X/K, then there exist $c, \epsilon > 0$ and a nonempty open subvariety U of X such that $h_{\overline{L}}(x) \geq \epsilon h_{\overline{M}}(\pi(x)) - c$ for all $x \in U(\overline{K})$.

Proof. For (1), because $\overline{L}$ is big, we may find some rational number $\epsilon > 0$ such that $\widehat{\mathrm{vol}}(\overline{L} - \epsilon\pi^*\overline{M}) > 0$ by Theorem 3.1.13. Therefore $h_{\overline{L}-\epsilon\pi^*\overline{M}} = h_{\overline{L}} - \epsilon h_{\overline{M}} \circ \pi$ is nonnegative on a dense open subvariety U of X. Namely, take $U = X - |\mathrm{div}(s)|$ where s is a nonzero effective section of (some multiple of) $\overline{L} - \epsilon\pi^*\overline{M}$, and we have the conclusion by the formula 3.1.1.

We now treat the case $k = \mathbf{Z}$, so that a quasiprojective model of X over $\mathbf{Z}$ has absolute dimension $d+1$. Let $f : X \to K$ be the structure map. If we choose $\overline{N} \in \widehat{\mathrm{Pic}}(K/\mathbf{Z})$ to have degree 1 (which is easily achieved by the explicit description in Example 2.3.39), we note that if $\overline{L}$ is nef on X/k, then with $\overline{L}' := \overline{L} + cf^*\overline{N}$, $\overline{L}'$ is also nef. Also, $f^*\overline{N}$ is nef (being a limit of pullbacks of a positive-degree Hermitian line bundles on $\mathrm{Spec}(\mathcal{O}_K)$), and so the arithmetic Hilbert-Samuel formula 3.1.12 gives

$$\mathrm{vol}(\overline{L}') = (\overline{L}')^{d+1} = \overline{L}^{d+1} + (d+1)c\overline{L}^d \cdot f^*\overline{N} \tag{3.1.2}$$

via the binomial expansion and the projection formula, and using the functoriality in [YZ24, Lemma 4.4.4(1), Lemma 4.6.1(2)], which implies that for Hermitian line bundles $\mathcal{L}$ and $\mathcal{N}$ approximating $\overline{L}$ and $\overline{N}$ on projective/integral models $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K) \to \mathrm{Spec}(\mathbf{Z})$, we have

$$\mathcal{L}^{d+1-k} \cdot (f^*\mathcal{N})^k = \mathcal{L}_\eta^{d+1-k} \cdot (f^*\mathcal{N}_\eta)^{k-1}$$

for $k \geq 1$, where the subscript η means restrict to the generic fiber of $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K)$. In particular if $k \geq 2$, this is 0 as $f^*\mathcal{N}_\eta$ is trivial. Now, the first term on the right-hand side of 3.1.2 is nonnegative by arithmetic Hilbert-Samuel again, and the second term is $(d+1)c\widetilde{L}^d$ by the same reasoning as above, which is positive by the bigness assumption. Therefore $\overline{L}'$ is big and we can apply part (1) to $\overline{L}'$ and $\overline{M}$, noting that the choice of $\overline{N}$ along with Remark 3.1.4 shows that $h_{\overline{L}'} = h_{\overline{L}} + c$.

For (3), we apply Lemma 3.1.14 to $\overline{L}$, so that there exists a rational number c such that $\overline{L} + cf^*\overline{N} \in \widehat{\mathrm{Pic}}(X/k)$ is big (where $\overline{N}$ is as above), and we conclude using part (1) again. □

3.2 Examples of big line bundles

The goal in this section will be to present some fundamental examples of big adelic line bundles. In this section the genus g of (relative) curves will always satisfy $g > 1$.

We need to first set up some definitions. Let k be $\mathbf{Z}$ or a field, S a quasiprojective *integral* scheme over k, and $\pi : X \to S$ a smooth relative curve with fibers of genus $g > 1$. We define a *stable compactification* of π to be a projective *integral* scheme $\overline{S}$ over k along with maps fitting in the diagram

$$\begin{array}{ccc} X & \hookrightarrow & \overline{X} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \overline{\pi}} \\ S & \hookrightarrow & \overline{S} \end{array}$$

with the horizontal arrows open immersions, and $\overline{\pi}$ a *stable* relative curve of genus g.[28] It is a fact that in our situation, a stable compactification always exists if we allow ourselves to pass to a finite cover S' of S, which can be taken to be normal, and replace π with the base change $\pi' : X_{S'} \to S'$. For the details see [Yua24, Section 3.1.4].

Let $\mathcal{M}_g$ be a fine moduli scheme of smooth curves of genus g over k, and $\pi_g : \mathcal{C}_g \to \mathcal{M}_g$ a universal curve over $\mathcal{M}_g$. Similarly let $\overline{\mathcal{M}}_g$ be a fine moduli scheme of stable curves of genus g over k, and $\overline{\pi}_g : \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$ a universal curve over $\overline{\mathcal{M}}_g$. To avoid the language of stacks (which I am not comfortable with), we may simply add a level-N structure to the moduli problem to get fine moduli spaces, but we omit this from the notation as it doesn't change any of the arguments. Set $\Delta \coloneqq \overline{\mathcal{M}}_g - \mathcal{M}_g$ the boundary divisor on $\overline{\mathcal{M}}_g$.

3.2.1 Hodge bundle

Our first example will be the Hodge bundle, which will play the role of the geometric object inducing the Faltings height.

Definition 3.2.1. The *Hodge bundle* on $\overline{\mathcal{M}}_g$, as a line bundle, is given by

$$\lambda_{\overline{\mathcal{M}}_g} \coloneqq \det(\pi_g)_*\omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g} = \bigwedge^g (\pi_g)_*\omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g}$$

where $\omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g}$ is a relative dualizing sheaf.

By cohomology and base change, the formation of Hodge bundle commutes with pullback. That is, for a stable relative genus-g curve $\pi : \overline{X} \to \overline{S}$ with smooth generic fiber, and $\iota : \overline{S} \to \overline{\mathcal{M}}_g$ the moduli morphism, the Hodge bundle

$$\lambda_{\overline{S}} \coloneqq \det \pi_*\omega_{\overline{X}/\overline{S}} = \bigwedge^g \pi_*\omega_{\overline{X}/\overline{S}}$$

[28] Recall that a connected projective (possibly non-integral) scheme of pure dimension 1 C over an algebraically closed field is called a stable curve if it is reduced, its singular points are ordinary double points, has arithmetic genus ≥ 2, and any rational irreducible component of C intersects other irreducible components (possibly itself) in at least 3 points. A stable relative curve then just means a relative curve whose geometric fibers are stable curves. More background can be found at [Liu02, Section 10.3].

is canonically isomorphic to $\iota^*\lambda_{\overline{\mathcal{M}}_g}$. Note that of course the definition of the Hodge bundle $\lambda_{\overline{S}}$ depends on $\overline{X}$ as well, but we make a heavy abuse of notation and omit it.

We now explain how to metrize this Hodge bundle in the case that $\pi : X \to S$ is smooth and S is a quasiprojective variety over $\mathbf{C}$. In this case, $\omega_{X/S}$ is simply $\Omega^1_{X/S}$, and for $s \in S(\mathbf{C})$, we can give the corresponding metric on the g-dimensional fiber $(\pi_*\omega_{X/S})(s) \cong \Gamma(X_s, \Omega^1_{X_s/s})$ to be

$$\langle \alpha, \beta \rangle := \frac{i}{2} \int_{X_s} \alpha \wedge \overline{\beta}$$

where α, β are holomorphic 1-forms on the Riemann surface X_s. The metric on λ_S is then simply the determinant of this metric (via the usual Gram-matrix construction on exterior powers of a vector space).

At this point it is useful to make a quick definition. We follow the normalization in [Yua24, Section 4.3.1].

Definition 3.2.2 ((Stable) Faltings height)**.** Let C be a geometrically integral, smooth, projective curve over K, where K is either a number field or a function field of one variable over k. Let K'/K be a finite extension such that C has semistable reduction over K'. We split into two parallel cases:

(1) If K is a number field, let $B' := \mathrm{Spec}(\mathcal{O}_{K'})$, and pick a stable relative curve $\mathcal{C}'/B'$ extending $C_{K'}/K'$ on the generic fiber, which is smooth. By the above discussion, we have a Hermitian line bundle $\overline{\lambda}_{B'}$ on B' with the metric at $\sigma \in K'(\mathbf{C})$ being the above metric of $\overline{\lambda}_{B',\sigma}$ on $\mathcal{C}'_\sigma/\mathbf{C}$. Then the *(stable) Faltings height* of C is

$$h_{Fal}(C) := \frac{1}{[K' : \mathbf{Q}]}\widehat{\deg}(\overline{\lambda}_{B'}).$$

It is called stable because it does not depend on the choice of K'.

(2) If K is a function field of one variable over k, let B' be the unique smooth projective curve over k with function field K'. Pick a stable relative curve $\mathcal{C}'/B'$ extending $C_{K'}/K'$ on the generic fiber, which is smooth. By the above discussion, we have a (vanilla) line bundle $\lambda_{B'}$ on B'. Then the *(stable) Faltings height* of C is

$$h_{Fal}(C) := \frac{1}{[K' : K]} \deg(\lambda_{B'})$$

where deg is the usual degree of divisors on k-curves. It is called stable because it does not depend on the choice of K', though it does depend on K.

There is another way to define the Hodge bundle. Let $\pi : X \to S$ be a *stable* relative curve of genus g. Then we may define the *relative Jacobian* $J := \mathrm{Pic}^0_{X/S}$, which is a smooth separated semiabelian group scheme over S [BLR90, Theorem 9.4.1]. If π is actually a *smooth* relative curve, then J is an *abelian* S-scheme [BLR90, Proposition 9.4.4]. In either case, if $e \in J(S)$ is the identity section, then define

$$\underline{\omega}_S \coloneqq \det(e^*\Omega^1_{J/S}) \cong e^*(\Omega^g_{J/S}).$$

Again in the smooth case we can metrize $\underline{\omega}_S$ when S is a quasiprojective variety over $\mathbf{C}$. Here for $s \in S(\mathbf{C})$, the metric on each fiber $\underline{\omega}_S(s) \cong e_s^*\Omega^g_{J_s/s} \cong \Gamma(J_s, \Omega^g_{J_s/s})$,[29] is

$$\langle \alpha, \beta \rangle \coloneqq \left(\frac{i}{2}\right)^g (-1)^{g(g-1)/2} \int_{J_s} \alpha \wedge \beta$$

where α, β are holomorphic g-forms on J_s.

These definitions are compatible:

Theorem 3.2.3. In the above notation, for $\pi : X \to S$ a smooth relative curve of genus g where S is a quasiprojective variety over $\mathbf{C}$, there is a canonical isometry of metrized line bundles $i : \lambda_S \xrightarrow{\sim} \underline{\omega}_S$. Moreover if the isomorphism of line bundles (without metrics) still holds if π is instead a stable relative curve.

Proof. By [BLR90, Theorem 8.4.1] the tangent space of J along the identity section is canonically identified with $R^1\pi_*\mathcal{O}_X$. By relative Serre duality we have $(R^1\pi_*\mathcal{O}_X)^\vee \cong \pi_*\omega_{X/S}$, so taking determinants we get the desired canonical isomorphism $\underline{\omega}_S \cong \lambda_S$.

Now we assume π is smooth and we prove that i is compatible with the metrics fiber-by-fiber over all $s \in S(\mathbf{C})$. For such s, we have

$$\lambda_S(s) = \det H^0(X_s, \Omega^1_{X_s/s}) \otimes_{\mathbf{C}} \left(\det H^1(X_s, \Omega^1_{X_s/s})\right)^{-1} = \det H^0(X_s, \Omega^1_{X_s/s})$$

by the definition and Serre duality. We would like to show that the metric

$$\left\|\omega_1^{X_s} \wedge \ldots \wedge \omega_g^{X_s}\right\|^2 = \det\left(\frac{i}{2}\int_{X_s} \omega_i^{X_s} \wedge \overline{\omega}_j^{X_s}\right)_{i,j} = \left(\frac{i}{2}\right)^g \det\left(\int_{X_s} \omega_i^{X_s} \wedge \overline{\omega}_j^{X_s}\right)_{i,j}$$

agrees with the metric

$$\left\|\omega_1^{J_s} \wedge \ldots \wedge \omega_g^{J_s}\right\|^2 = \left(\frac{i}{2}\right)^g (-1)^{g(g-1)/2} \int_{J_s} \omega_1^{J_s} \wedge \ldots \wedge \omega_g^{J_s} \wedge \overline{\left(\omega_1^{J_s} \wedge \ldots \wedge \omega_g^{J_s}\right)}.$$

In the following we will use some facts about the construction of the Jacobian variety from [Mil86, Section 5]. Writing X (resp. J) for X_s (resp. J_s), if we choose a point $P \in X(\mathbf{C})$,

[29] Here the second isomorphism follows since holomorphic differentials on an abelian variety are determined by their value at the identity [Mum08, (iii), pg. 39].

then we have a map $\pi : X^g \to J$ given by the g-fold sum of the Abel–Jacobi embeddings $\pi_1, \ldots, \pi_g$ each with basepoint P, which is finite of degree $g!$. Moreover, for notational purposes, if $H^0(X, \Omega^1_{X/\mathbf{C}}) \simeq H^0(J, \Omega^1_{J/\mathbf{C}})$ is the isomorphism induced from the Abel–Jacobi embedding, then we denote by ω^X and ω^J corresponding 1-forms on the two sides of this isomorphism. In particular $\pi^*\omega_i^J = \sum_{j=1}^g \pi_j^*\omega_i^J$ and

$$\int_X \pi_i^*\omega_j^J \wedge \pi_i^*\overline{\omega}_k^J = \int_X \omega_j^X \wedge \overline{\omega}_k^X.$$

Hence pulling back via π, we get

$$\begin{aligned}\int_J \omega_1^J \wedge \ldots \omega_g^J \wedge \overline{\left(\omega_1^J \wedge \ldots \wedge \omega_g^J\right)} &= \frac{1}{g!}\int_{X^g} \pi^*\omega_1^J \wedge \ldots \wedge \pi^*\overline{\omega_g^J} \\ &= \frac{1}{g!}\int_{X^g} \left(\sum_{j=1}^g \pi_j^*\omega_1^J\right) \wedge \ldots \wedge \left(\overline{\sum_{j=1}^g \pi_j^*\omega_g^J}\right) \\ &= (-1)^{g(g-1)/2}\frac{1}{g!}\sum_{\tau\in S_g}\sum_{\sigma\in S_g}\operatorname{sgn}(\tau)\operatorname{sgn}(\sigma)\prod_{j=1}^g \int_{X_j} \pi_j^*\omega_{\tau(j)}^J \wedge \pi_j^*\overline{\omega_{\sigma(j)}^J}\end{aligned}$$

by Fubini's theorem, where X_j denotes the jth copy of the Riemann surface X inside X^g. Note that the coefficient $(-1)^{g(g-1)/2}$ appears because we need to rearrange a differential form of the form

$$\pi_1^*\omega_{\tau(1)}^J \wedge \pi_2^*\omega_{\tau(2)}^J \wedge \ldots \wedge \pi_{g-1}^*\overline{\omega_{\sigma(g-1)}^J} \wedge \pi_g^*\overline{\omega_{\sigma(g)}^J}$$

into the form

$$\pi_1^*\omega_{\tau(1)}^J \wedge \pi_1^*\overline{\omega_{\sigma(1)}^J} \wedge \ldots \wedge \pi_g^*\omega_{\tau(g)}^J \wedge \pi_g^*\overline{\omega_{\sigma(g)}^J}$$

in order to apply Fubini's theorem. This is a "perfect (riffle) shuffle" permutation on $2g$ elements, which is easily seen (by induction) to have sign $(-1)^{\lfloor g/2 \rfloor} = (-1)^{g(g-1)/2}$. Then

$$\begin{aligned}(-1)^{g(g-1)/2}\frac{1}{g!}\sum_{\tau\in S_g}\sum_{\sigma\in S_g}\operatorname{sgn}(\tau)\operatorname{sgn}(\sigma)\prod_{j=1}^g \int_{X_j} \pi_j^*\omega_{\tau(j)}^J \wedge \pi_j^*\overline{\omega_{\sigma(j)}^J} &= (-1)^{g(g-1)/2}\frac{1}{g!}\sum_{\tau\in S_g}\operatorname{sgn}(\tau)\sum_{\sigma\in S_g}\operatorname{sgn}(\tau\sigma)\prod_{j=1}^g \int_{X_j} \omega_{\tau(j)}^{X_j} \wedge \overline{\omega_{\tau\sigma(j)}^{X_j}} \\ &= (-1)^{g(g-1)/2}\frac{g!}{g!}\sum_{\sigma\in S_g}\operatorname{sgn}(\sigma)\prod_{j=1}^g \int_{X_j} \omega_j^{X_j} \wedge \overline{\omega_{\sigma(j)}^{X_j}} \\ &= (-1)^{g(g-1)/2}\det\left(\int_X \omega_i^X \wedge \overline{\omega}_j^X\right)_{i,j}.\end{aligned}$$

□

The above argument follows Szpiro [Szp85, Lemma 3.2.1], except we have paid attention to the signs.

In the arithmetic case, for instance when $S = \operatorname{Spec}(\mathcal{O}_K)$, it is now possible to define the *Hodge bundle* as an *adelic line bundle* on S. Unfortunately the construction is quite technical, so we will only state the result and a reference.

Theorem 3.2.4. [YZ24, Theorem 5.5.1] Let S be a flat quasiprojective integral scheme over $\mathbf{Q}$ or $\mathbf{Z}$, $\pi : X \to S$ a smooth relative curve of genus g, and $(\lambda_S, \|\cdot\|)$ the (metrized) Hodge bundle on S as constructed above. Then there is a canonically defined adelic line bundle $\overline{\lambda}_S \in \widehat{\mathrm{Pic}}(S/\mathbf{Z})$ which extends $(\lambda_S, \|\cdot\|)$. By this we mean the underlying line bundle of $\overline{\lambda}_S$ is λ_S, and the metric on $S(\mathbf{C})$ induced by $\overline{\lambda}_S$ is exactly $\|\cdot\|$.

Moreover, the (adelic) height (via Definition 3.1.1) $h_{\overline{\lambda}_S}(s)$ for $s \in S(\overline{\mathbf{Q}})$ is precisely $h_{Fal}(X_s)$, where h_{Fal} is the stable Faltings height of the curve X_s as introduced in Definition 3.2.2.

We also need the following theorem to discuss the geometric versions of the main theorems. The proof can be found on [Yua24, pgs. 55–56], which uses the λ_S version of the Hodge bundle instead the $\underline{\omega}_S$ version.

Theorem 3.2.5. Suppose S is a quasiprojective normal integral scheme over a *field* k. Let $\pi : X \to S$ be a smooth relative curve of genus g with maximal variation, meaning that the moduli map from S into the coarse moduli scheme $M_{g,k}$ is generically finite, and $\overline{\pi} : \overline{X} \to \overline{S}$ a stable compactification. Then the (vanilla) line bundle $\lambda_{\overline{S}}$ is nef and big on $\overline{S}$.

Note that since $\overline{S}$ is a projective model of S by assumption and we are in the geometric case (i.e. no Hermitian metrics), $\lambda_{\overline{S}}$ determines an adelic line bundle of $\widehat{\mathrm{Pic}}(S/k)$ by the constant sequence $(\lambda_S, (\overline{S}, \lambda_{\overline{S}}, \mathrm{id}))$. Therefore we may, by abuse of notation, consider $\lambda_{\overline{S}} \in \mathrm{Pic}(\overline{S})$ as an adelic line bundle in $\widehat{\mathrm{Pic}}(S/k)$. Then $\lambda_{\overline{S}} \in \widehat{\mathrm{Pic}}(S/k)$ is nef and big by Definition–Theorem 3.1.8. We will appeal to this fact in the next section.

Example 3.2.6. To finish, we write out the height associated to $\lambda_{\overline{S}}$ in the geometric case, since we will use this below. Let K be a function field of one variable over a field k, and let S now be a quasiprojective variety over k. Let $\pi : X \to S$ be a smooth relative curve of genus g, and assume for simplicity that it has a stable compactification $\overline{\pi} : \overline{X} \to \overline{S}$. Recall from Definition 3.2.2 that the stable Faltings height of X_s for $s \in S(\overline{K}) = S_K(\overline{K})$ is defined as follows:

$$h_{Fal}(X_s) = \frac{1}{[K' : K]} \deg_{B/k}(\lambda_B)$$

where K' is an extension of K such that X_s has stable reduction over K', B is the unique smooth projective curve over k with function field K', and λ_B is the Hodge bundle on B of a stable relative curve $\mathcal{C} \to B$ extending $X_s \to K'$ on the generic fiber. But by the assumptions and the valuative criterion, we may take $K' = K(s)$ and the map $s : K(s) \to \overline{S}$ extends to a map $B \to \overline{S}$. Then $\lambda_{\overline{S}}|_s \in \widehat{\mathrm{Pic}}(K(s)/k)$ is represented by the sequence $(\lambda_s, (B, \lambda_B, \mathrm{id}))$ where the Hodge bundle λ_B on B comes from $\mathcal{C} = \overline{X}_B \to B$ via functoriality.

On the other hand, from Definition 3.1.1 we need to first base change to S_K in order to define a height coming from an adelic line bundle in $\widehat{\mathrm{Pic}}(S_K/k)$. Here S_K is essentially quasiprojective over k, so we can pullback $\lambda_{\overline{S}} \in \widehat{\mathrm{Pic}}(S/k)$ to an element of $\widehat{\mathrm{Pic}}(S_K/k)$ via the procedure laid out in [YZ24, Section 2.5.5] and sketched after Definition 2.3.40. To recall that, let $\mathcal{U}$ be a quasiprojective model of S_K with a map to S over k, and by the discussion in loc. cit. there is a projective model $\mathcal{X}$ of $\mathcal{U}$ along with a morphism $f : \mathcal{X} \to \overline{S}$ extending $f_S : \mathcal{U} \to S$. The construction shows that the pullback is represented by the sequence $(f_S^*\lambda_S, (\mathcal{X}, f^*\lambda_{\overline{S}}, \mathrm{id}))$. Then the restriction of this pullback to $s \in S_K(\overline{K}) \subseteq U(\overline{K}) \subseteq \mathcal{X}(\overline{K})$ is precisely $(\lambda_s, (B, \lambda_B, \mathrm{id}))$, so that

$$h_{\lambda_{\overline{S}}}(s) = \frac{\widehat{\deg}\langle \lambda_{\overline{S}}|_s \rangle}{[K(s) : K]} = h_{Fal}(X_s).$$

Above on the left-hand side, we consider $\lambda_{\overline{S}} \in \widehat{\mathrm{Pic}}(S_K/k)$ via the above discussion.

3.2.2 Globalized self-intersection of the dualizing sheaf

In this section k always denotes a field.

The really key example will be the following:

Theorem 3.2.7. Let S a quasiprojective normal integral scheme over k. Let $\pi : X \to S$ be a smooth relative curve[30] of genus $g > 1$ with maximal variation. Then the Deligne pairing $\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle$ is big and nef on S.

There is a number field analogue of this, which is described in [Yua24, Section 3.5]. On the other hand for simplicity we only work in the function field case.

Note that the nefness is actually immediate because $\overline{\omega}_{X/S,a}$ is nef by Theorem 2.3.58, and the Deligne pairing of nef adelic line bundles is nef as mentioned in Definition 2.3.46. On the other hand, it will be the bigness that is important for the application of the height inequality. First, we give a summary of the proof. The key claim is the following ([Yua24, Theorem 3.8]):

Theorem 3.2.8. Let S be a quasiprojective normal integral scheme over k, and let $\pi : X \to S$ be a smooth relative curve of genus $g > 1$ with a stable compactification $\overline{\pi} : \overline{X} \to \overline{S}$. In $\widehat{\mathrm{Pic}}(S)_{\mathbf{Q}}$, we have an equality

$$\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle = \left(1 + \frac{39(3g-1)(2g-1)}{2}\right) \lambda_{\overline{S}} + \overline{A} + \mathcal{O}(\overline{B})$$

where $\overline{A} \in \widehat{\mathrm{Pic}}(S)_{\mathbf{Q}}$ is nef and $\overline{B} \in \widehat{\mathrm{Div}}(S)$ is an effective adelic divisor with underlying divisor $0 \in \mathrm{Div}(S)$.

[30]Note that this automatically implies X is integral, and even normal, so the Deligne pairing of adelic line bundles on X makes sense.

Theorem 3.2.8 implies Theorem 3.2.7. First we show that the problem is unaffected upon passing to a finite normal cover $f : S' \to S$. Then bigness on S is equivalent to bigness of the pullback on S' by the projection formula Theorem 2.3.45, since the volume can be computed via the top self-intersection number in the nef case (which we assume is already proved) due to the Hilbert–Samuel formula Theorem 3.1.12. In particular, $\pi_*\langle\overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a}\rangle$ is big if and only if $f^*\pi_*\langle\overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a}\rangle \cong \pi'_*\langle\overline{\omega}_{X_{S'}/S',a}, \overline{\omega}_{X_{S'}/S',a}\rangle$ is, since the Deligne pairing is compatible with pullback as noted in Definition-Theorem 2.3.46.

The point of passing to this finite normal cover is so that we can have a stable compactification and apply Theorem 3.2.8. Having replaced S and X by S' and $X_{S'}$ respectively, we just need to prove that $\left(1 + \frac{39(3g-1)(2g-1)}{2}\right)\lambda_{\overline{S}} + \overline{A} + \mathcal{O}(\overline{B})$ is big. The sum $\left(1 + \frac{39(3g-1)(2g-1)}{2}\right)\lambda_{\overline{S}} + \overline{A}$ is the sum of two nef adelic line bundles, hence nef. Since the first term is big by Theorem 3.2.5 (*and this is the only place the maximal variation hypothesis is used in the proof of Theorem 3.2.7*), the sum is big via the arithmetic Hilbert–Samuel formula (binomial expansion of the top self-intersection number gives a sum of a positive leading term and other nonnegative terms). Then we just need to show that the sum of a big adelic line bundle and an effective adelic line bundle is big. More generally, if $\overline{L}, \overline{M}$ are adelic line bundles on Y such that $\overline{M} - \overline{L}$ is effective, then $\widehat{\mathrm{vol}}(Y, \overline{L}) \leq \widehat{\mathrm{vol}}(Y, \overline{M})$. Exactly as in the classical setting, this is because for a nonzero effective section $s \in \widehat{H}^0(Y, \overline{M} - \overline{L})$, tensoring against s gives an injection $\widehat{H}^0(Y, \overline{L}) \hookrightarrow \widehat{H}^0(Y, \overline{M})$, and hence $\widehat{h}^0(Y, n\overline{L}) \leq \widehat{h}^0(Y, n\overline{M})$ for all $n \in \mathbf{N}$. □

The proof of Theorem 3.2.8 is achieved by the following steps:

(1) We first use the existence of the stable compactification $\overline{\pi} : \overline{X} \to \overline{S}$ to write

$$\pi_*\langle\overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a}\rangle = \overline{\pi}_*\langle\omega_{\overline{X}/\overline{S}}, \omega_{\overline{X}/\overline{S}}\rangle - \mathcal{O}(\overline{E}_S) \tag{3.2.1}$$

inside $\widehat{\mathrm{Pic}}(S)_{\mathbf{Q}}$, where $\overline{E}_S$ is an effective adelic divisor on S with underlying divisor 0. Here the line bundle $\overline{\pi}_*\langle\omega_{\overline{X}/\overline{S}}, \omega_{\overline{X}/\overline{S}}\rangle$ on $\overline{S}$ is treated as an adelic line bundle on S in the same manner as above. This should be viewed as a "globalization/relativization" of Zhang's theorem for a single curve C over a global field K [Zha93, Theorem 5.5], which follows from the local case that we presented in Theorem 2.3.15.

$$\langle\overline{\omega}_{C/K,a}, \overline{\omega}_{C/K,a}\rangle = \langle\omega_{C/K,Ar}, \omega_{C/K,Ar}\rangle \otimes \mathcal{O}\left(\sum_{v\in M_K} \epsilon_v[v]\right)^{-1} \tag{3.2.2}$$

Here ϵ_v is Zhang's *ϵ-invariant* of the *reduction graph* associated to a stable model of C over a valuation ring of K_v, as we introduced in Theorem 2.3.15. Recall that $\epsilon_v \geq 0$ with equality if and only if C has good reduction at v, or if v is an infinite place. Note that in the function field case (i.e. when K is a field of transcendence degree 1 over a base field k), there are no infinite places.

Remark 3.2.9. Here is an important but unnecessary (for us) remark. As mentioned in the introduction, the Bogomolov conjecture for a single curve C is implied once $\omega^2_{C/K,Ar} > 0$ [Szp90, Theorem 3]. Zhang shows [Zha93, Corollary 5.7(1)] that $\overline{\omega}^2_{C/K,a} \geq 0$, so upon taking degrees in Equation 3.2.2 we get the Bogomolov conjecture for curves that have at least one finite place of bad reduction.

(2) By the "universal" Noether formula due to Mumford [Mum77, Theorem 5.10, pg. 102],

$$12\lambda_{\overline{\mathcal{M}}_g} = (\overline{\pi}_g)_*\langle \omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g}, \omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g} \rangle + \mathcal{O}_{\overline{\mathcal{M}}_g}(\Delta),$$

we can pull back to $\pi : \overline{X} \to \overline{S}$:

$$12\lambda_{\overline{S}} = \overline{\pi}_*\langle \omega_{\overline{X}/\overline{S}}, \omega_{\overline{X}/\overline{S}} \rangle + \mathcal{O}(\Delta_{\overline{S}}). \tag{3.2.3}$$

Here $\Delta_{\overline{S}}$ is defined as the pullback of the boundary divisor Δ on $\overline{\mathcal{M}}_g$ to $\overline{S}$. The intuition is that it is a "globalization" of the δ-*invariant* of $\overline{X}/\overline{S}$. In general, for a single curve C over a global field K with *regular* (minimal) semistable model $\mathcal{C}$, we define δ_v for a place $v \in M_K$ as the number of singularities of the fiber of $\mathcal{C}$ over v if v is a finite place, and the Faltings' delta-invariant of the Riemann surface $\mathcal{C}_v$ if v is infinite (as introduced in [Fal84]). Once again, in our function field case, there are no infinite places, which makes the situation significantly simpler as it is a difficult problem to bound the Archimedean delta-invariants.

As before, the above equality is of $\mathbf{Q}$-line bundles in $\mathrm{Pic}(\overline{S})_{\mathbf{Q}}$, but we can view everything in $\widehat{\mathrm{Pic}}(S/k)_{\mathbf{Q}}$ as well.

(3) Once the "globalizations" have been completed, we can compare the adelic divisors $\overline{E}_S, \Delta_{\overline{S}} \in \widehat{\mathrm{Div}}(S/k)$ using "local" graph-theoretic computations on single curves (the point of these globalizations was to assemble these local invariants into a single geometric object for any given family of curves). Such a graph-theoretic computation shows that $(2g-2)\Delta_{\overline{S}} - \overline{E}_S$ is an effective adelic divisor in $\widehat{\mathrm{Div}}(S/k)$. From Equations 3.2.1 and 3.2.3 we now have

$$\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle = 12\lambda_{\overline{S}} - (2g-1)\mathcal{O}(\Delta_{\overline{S}}) + \mathrm{eff} \tag{3.2.4}$$

inside $\widehat{\mathrm{Pic}}(S/k)$, where "eff" denotes some effective adelic line bundle.

Notice that at this point we are almost done, because we have already proved the bigness of $\lambda_{\overline{S}}$ above. The issue is of course the $-(2g-1)\mathcal{O}(\Delta_{\overline{S}})$ term, which we need to control. The goal is then to show that there is some large multiple of $\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle$ which is "larger" than $\mathcal{O}(\Delta_{\overline{S}})$ (of course uniformly for all $\pi : X \to S$), where "larger" means that the difference is a sum of some positive line bundles (can be nef, effective, etc.). In that way some multiple of $\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle$ will be the desired sum of a big line bundle and other "positive" line bundles.

(4) As mentioned above, we essentially need a "globalized/relativized lower bound" of the self-intersection of $\overline{\omega}_{X/S,a}$. For this, we introduce yet another graph-theoretic invariant, Zhang's *φ-invariant* as introduced in [Zha10, Section 4.1]. It plays a role in Cinkir's resolution of the effective Bogomolov conjecture in function fields of characteristic 0, which we will say a bit more on in the next item. For now, the proof of [dJ18, Theorem 8.1] shows that for a single curve C over a global field K, there is an admissible adelic line bundle $\overline{M} \coloneqq 2\overline{\mathcal{O}}(\Delta)_a + p_1^*\overline{\omega}_{C/K,a} + p_2^*\overline{\omega}_{C/K,a}$ on C^2, such that

$$\overline{M}^3 = (12g-4)\overline{\omega}^2_{C/K,a} - 8\sum_{v\in M_K}\varphi(C_v)\log(n_v). \tag{3.2.5}$$

To explain the notations, in the definition of $\overline{M}$, p_1, p_2 are the two projections $C\times C \rightrightarrows C$, Δ is the diagonal divisor in C^2, and $\mathcal{O}(\Delta)_a$ is an admissible adelic line bundle with underlying line bundle $\mathcal{O}(\Delta)$. This $\mathcal{O}(\Delta)_a$ is constructed in [Zha93, Section 4.7]. Finally, $\varphi(C_v)$ is the φ-invariant of the reduction graph associated to a stable model of C over a valuation ring of K_v.[31]

Thus in this step we globalize/relativize de Jong's result exactly as in Step 1. Namely, there is an effective adelic divisor $\overline{\Phi}_S$ on S with underlying divisor 0, such that with $(\pi,\pi) : X\times_S X \to X$ the structure map,

$$(\pi,\pi)_*\langle\overline{M},\overline{M},\overline{M}\rangle = (12g-4)\pi_*\langle\overline{\omega}_{X/S,a},\overline{\omega}_{X/S,a}\rangle - 8\mathcal{O}(\overline{\Phi}_S). \tag{3.2.6}$$

Moreover, it is proved that $(\pi,\pi)_*\langle\overline{M},\overline{M},\overline{M}\rangle$ is *nef*. In particular in $\widehat{\mathrm{Pic}}(S)_{\mathbf{Q}}$,

$$\pi_*\langle\overline{\omega}_{X/S,a},\overline{\omega}_{X/S,a}\rangle = \frac{2}{3g-1}\mathcal{O}(\overline{\Phi}_S) + \text{nef}. \tag{3.2.7}$$

(5) Cinkir proved the effective Bogomolov conjecture over function fields of characteristic 0 by finding an explicit lower bound on $\overline{\omega}^2_{C/K,a}$ via Zhang's bound [Zha10, pg. 10]

$$\overline{\omega}^2_{C/K,a} \geq \frac{2g-2}{2g+1}\sum_{v\in M_K}\varphi(C_v)\log(n_v).$$

This allows one to explicitly compute a value for the "ϵ" in the Bogomolov conjecture, as explained in [Szp90, Section 2.1].

Specifically, [Cin11, Theorem 2.11] proves that for any $v \in M_K$,

$$\varphi(C_v) \geq c(g)\delta_0(C_v) + \sum_{i=1}^{\lfloor g/2\rfloor}\frac{2i(g-i)}{g}\delta_i(C_v)$$

[31] If we were working in the number field case and v was an infinite place of K, there would be a different definition. The interested reader can refer to [Zha10, Proposition 2.5.3].

where δ_i, $0 \leq i \leq \lfloor g/2 \rfloor$ are refined singularity-counting functions of the fiber C_v whose sum is δ_v (as defined in Step 2), and $c(2) = 1/27$, $c(g) = \frac{(g-1)^2}{2g(7g+5)}$ for $g \geq 3$. In particular $\varphi(C_v) \geq \delta_v/39$, and the global version of this is that

$$\overline{\Phi}_S = \frac{1}{39}\Delta_{\overline{S}} + \mathrm{eff} \tag{3.2.8}$$

in $\widehat{\mathrm{Div}}(S)_{\mathbf{Q}}$.

Finally, combining Equations 3.2.7 and 3.2.8 we obtain

$$\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle = \frac{2}{39(3g-1)}\mathcal{O}(\Delta_{\overline{S}}) + \mathrm{nef} + \mathrm{eff}. \tag{3.2.9}$$

This is the lower bound on $\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle$ in terms of $\frac{2}{39(3g-1)}\mathcal{O}(\Delta_{\overline{S}})$ that we wanted at the end of Step 3. Combining this with Equation 3.2.4 we get

$$\left(1 + \frac{39(3g-1)(2g-1)}{2}\right)\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle = 12\lambda_{\overline{S}} + \mathrm{nef} + \mathrm{eff}$$

inside $\widehat{\mathrm{Pic}}(S)_{\mathbf{Q}}$. Moreover, it is clear from the above description that the effective adelic $\mathbf{Q}$-divisor in the above equation was only obtained from linear combinations of the "globalizing" adelic divisors $\Delta_{\overline{S}}, \overline{E}_S, \overline{\Phi}_S$, each of which have trivial underlying divisor by construction (they are only nontrivial in their analytic components, which record the graph-theoretic data as described above). This proves Theorem 3.2.8.

Having sketched the proof, we now treat each step in more detail.

Step 1: From Theorem 2.3.58 we recall the construction of the admissible adelic line bundle $\overline{\omega}_{X/S,a} \in \widehat{\mathrm{Pic}}(X/k)$. By definition, it has underlying line bundle $\omega_{X/S}$, the metric of $\omega_{X_{H_v}/H_v}$ on $X_{H_v}^{an}$ is the canonical admissible metric (where H_v is the completed residue field at v), and it is uniquely characterized by these conditions. We can also restrict $\omega_{\overline{X}/\overline{S}} \in \mathrm{Pic}(\overline{S}/k)$ to an adelic line bundle in $\widehat{\mathrm{Pic}}(S/k)$ by taking the underlying line bundle to be $\omega_{X/S}$ and a constant sequence of model adelic line bundles, just as we did with the Hodge bundle in the above Section 3.2.1. Since the Deligne pairing is constructed via representing sequences of model adelic line bundles (cf. [YZ24, Section 4.5]), we see that

$$\overline{\pi}_*\langle \omega_{\overline{X}/\overline{S}}, \omega_{\overline{X}/\overline{S}} \rangle - \pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle =: \mathcal{O}(\overline{E}_S) \tag{3.2.10}$$

in $\widehat{\mathrm{Pic}}(S/k)$ for some adelic line bundle $\mathcal{O}(\overline{E}_S)$ with underlying line bundle canonically isomorphic to $\mathcal{O}_S$. We may then take $\overline{E}_S \in \widehat{\mathrm{Div}}(S/k)$ to be $\widehat{\mathrm{div}}(t)$, where t is a rational section of underlying line bundles of projective models of $\overline{\pi}_*\langle \omega_{\overline{X}/\overline{S}}, \omega_{\overline{X}/\overline{S}} \rangle - \pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle$ corresponding to $1 \in \mathcal{O}_S$ under said isomorphism. We implicitly defined the construction of

$\widehat{\mathrm{div}}$ in Section 2.3.3, right before Definition 2.3.36. Note that $\mathrm{div}(t) = 0$ on S, i.e. $\overline{E}_S$ has underlying divisor 0.

To explicitly describe $\overline{E}_S$, it suffices to describe its analytification in $\widehat{\mathrm{Div}}(S^{an})_{eqv}$ by Theorem 2.3.56, which is done once we specify its (continuous) Green's function $g_{\overline{E}_S} : S^{an} \to \mathbf{R}$ (the underlying divisor of $\overline{E}_S$ is 0, so the Green's function is defined everywhere). By [YZ24, Lemma 3.1.1], the set of $v \in S^{an}$ corresponding to discrete valuations[32] of $K(S)$ (via the natural map $K(S)^{an} \to S^{an}$) is dense in S^{an},[33] so we only need to specify $g_{\overline{E}_S}(v)$ at such v.

By the norm-equivariance property, we may assume that the discrete valuation v of $K(S)$ has $|\pi_v|_v = 1/e$ ($e = 2.718\ldots$), where π_v is a uniformizer for v. Consider the equality (3.2.10) after taking the analytification of both sides, so both sides are metrized line bundles over S^{an}. The right-hand side has trivial underlying line bundle, but we want to understand the metric at v. To do this, consider the map $\mathrm{Spec}(\mathcal{O}_{H_v}) \to \overline{S}$ obtained from $\mathrm{Spec}(H_v) \to \overline{S}$ via the valuative criterion, and let $X_{\mathcal{O}_{H_v}} \to \mathrm{Spec}(\mathcal{O}_{H_v})$ be the base change of $\overline{\pi} : \overline{X} \to \overline{S}$. Also let X_{H_v} and X_v be the generic and special fibers of $X_{\mathcal{O}_{H_v}}$, respectively. Recall that the Deligne pairing and the formation of the admissible relative dualizing sheaf are compatible with base change (the latter by the uniqueness in Theorem 2.3.58). Then over the fiber at v (i.e. metrized H_v-vector spaces), we have by Theorem 2.3.15:

$$\begin{aligned}\overline{\pi}_*\langle \omega_{\overline{X}/\overline{S}}, \omega_{\overline{X}/\overline{S}}\rangle^{an}|_{H_v} - \pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a}\rangle^{an}|_{H_v} &= \langle \omega_{X_{H_v}/H_v}, \omega_{X_{H_v}/H_v}\rangle - \langle \omega_{X_{H_v}/H_v,a}, \omega_{X_{H_v}/H_v,a}\rangle \\ &= \mathcal{O}(\epsilon_v).\end{aligned}$$

To recall the terms, $\langle \omega_{X_{H_v}/H_v}, \omega_{X_{H_v}/H_v}\rangle - \langle \omega_{X_{H_v}/H_v,a}, \omega_{X_{H_v}/H_v,a}\rangle$ should be interpreted in the sense of the local Deligne pairing as explained in [YZ24, Section 4.6.2]. Also, $\mathcal{O}(\epsilon_v)$ is the metrized 1-dimensional vector space $(H_v, \|\cdot\|)$ on H_v where $\|1\| = \exp(-\epsilon_v)$ and

$$\epsilon_v := \int_{\Gamma_v} g_\mu(x,x)((2g-2)\mu + \delta_{K_{X_v}}).$$

In the definition of ϵ_v, Γ_v is the polarized metrized reduction graph of X over $k(v)$,[34] [35]

$$K_{X_v} := \sum_{p \in V(\Gamma_v)} (v(p) - 2 + 2q(p))[p]$$

[32] By "discrete valuation" we really mean the corresponding absolute value.

[33] Here there are no Archimedean valuations as we are working in the geometric case, where k always has the trivial valuation as per our conventions.

[34] See Section 2.3.1 and Definition 2.3.9 for the definition and conventions on the reduction graph.

[35] As a small detail, we may need to take a finite extension of $k(v)$ in order to ensure all nodes of X_v are rational over $k(v)$, since we want this condition for the reduction graph (per the conventions in [YZ24, Appendix A.5]) but we did not include it in the definition of (semi)stable curves. This can be achieved by taking a finite unramified base change of $\mathcal{O}_{H_v}$, which is harmless.

is the canonical divisor on the polarized metrized graph Γ_v, and μ is the canonical admissible metric associated to K_{X_v} via Theorem 2.3.12. Therefore the Green's function of $\overline{E}_S^{an} = \widehat{\operatorname{div}}_{\overline{S}^{an}}(t) = (\operatorname{div}(t), -\log\|t\|)$ at v is precisely

$$g_{\overline{E}_S}(v) = -\log\|t(v)\| = \epsilon_v$$

because the isometry $\mathcal{O}(\overline{E}_S)^{an}|_{H_v} \xrightarrow{\sim} \mathcal{O}(\epsilon_v)$ of metrized H_v-line bundles sends t to 1.[36]

In general, by the norm-equivariance property, if v' is a discrete valuation of $K(S)$ such that locally on S^{an}, $v' = v^r$ for $r \in [0, \infty)$ as above, then

$$g_{\overline{E}_S}(v') = r g_{\overline{E}_S}(v) = g_{\overline{E}_S}(v) \cdot \log\left(|\pi_{v'}|_{v'}^{-1}\right),$$

where $\pi_{v'} = \pi_v$ is a uniformizer for both v' and v.

Step 2: Recall the universal Noether formula

$$12\lambda_{\overline{\mathcal{M}}_g} = (\overline{\pi}_g)_*\langle \omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g}, \omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g}\rangle + \mathcal{O}_{\overline{\mathcal{M}}_g}(\Delta),$$

which we can pull back to $\pi : \overline{X} \to \overline{S}$:

$$12\lambda_{\overline{S}} = \overline{\pi}_*\langle \omega_{\overline{X}/\overline{S}}, \omega_{\overline{X}/\overline{S}}\rangle + \mathcal{O}(\Delta_{\overline{S}}). \tag{3.2.11}$$

Here $\Delta_{\overline{S}}$ is defined as the pullback of the boundary divisor Δ on $\overline{\mathcal{M}}_g$ to $\overline{S}$. Treating these as adelic line bundles over S, we have an equality in $\widehat{\operatorname{Pic}}(S/k)$.

We would like to give a more explicit description of $\Delta_{\overline{S}}$ in terms of its analytification. From the description in [CH88, Section 4a], we have that Δ is a union of irreducible divisors $\Delta_0, \Delta_1, \ldots, \Delta_{\lfloor g/2\rfloor}$ in $\overline{\mathcal{M}}_g$, where Δ_i for $i > 0$ (resp. $i = 0$) parameterizes stable curves C such that the *partial* normalization of C at one of its nodes consists of two connected components of arithmetic genera i and $g-i$ (resp. is connected). Such a node will be called "type i," and Δ_i pull back to divisors $\Delta_{\overline{S},i}$ on $\overline{S}$. Now suppose $s \in \overline{S}$ and $p_1, \ldots, p_{j(i)}$ are the nodes of type i in $\overline{X}_s$, so (analytic-)locally near p_k, $\overline{X}$ is of the form $xy = \gamma_{i,k}$ where $\gamma_{i,k}$ is a nonzero local function on S that vanishes at s. Therefore $\Delta_{\overline{S},i}$ is locally defined by $f_i \coloneqq \prod_{k=1}^{j(i)} \gamma_{i,k}$ near s.

As in Step (1), to describe the Green's function $g_{\Delta_{\overline{S}}}$ of the analytification, we only need to describe what it is on discrete valuations v of $K(S)$, and we can also assume that v is normalized by $|\pi_v|_v = 1/e$. Consider $r(v) \in U \subseteq \overline{S}$, where U is an affine open containing $r(v)$ (recall that $r(v)$ is defined to be the image of the closed point under the natural map $\operatorname{Spec}(\mathcal{O}_{H_v}) \to \overline{S}$, cf. Definition 2.3.50). By definition,

$$g_{\Delta_{\overline{S}}}(v) = -\log\left|\prod_{i=0}^{\lfloor g/2\rfloor} f_i\right|_v = \sum_{i=0}^{\lfloor g/2\rfloor} \operatorname{ord}_v(f_i).$$

[36]Note that the image of $\operatorname{Spec}(H_v) \to \overline{S}$ actually lands in S, so t trivializes the underlying line bundle of $\mathcal{O}(\overline{E}_S)^{an}$ at H_v.

where the $f_i \in \mathcal{O}_{H_v}$ locally cut out $\Delta_{\overline{S},i}$ at $r(v)$ as in the notation of the previous paragraph.

We claim that $g_{\Delta_{\overline{S}}}(v) = \delta_v$, where δ_v is the number of edges in Γ_v, or equivalently the number of singular points in the fiber X_v of a *regular* (minimal) semistable model of X over $\mathcal{O}_{H_v}$ (by abuse of notation we also use X for the base change to $\mathcal{O}_{H_v}$). To find such a regular model, we blow up the non-regular points in the special fiber, which are exactly the nodal points p where X has local equation $xy = \gamma$, $\mathrm{ord}_v(\gamma) \coloneqq n > 1$. After repeated blowing-ups until X becomes regular, the preimage of p is a chain of $n-1$ rational curves defined over $k(v)$ that cross transversally at rational points (necessarily with local equation $xy = \pi$ where π is a uniformizer of $\mathcal{O}_{H_v}$, by regularity), by [Liu02, Corollary 10.3.25]. In particular, the preimage of each node p contains $n = \mathrm{ord}_v(\gamma)$ singular points, and so the reduction graph Γ_v indeed contains

$$\delta_v = \sum_{i=0}^{\lfloor g/2 \rfloor} \sum_{k=1}^{j(i)} \mathrm{ord}_v(\gamma_{i,k}) = \sum_{i=0}^{\lfloor g/2 \rfloor} \mathrm{ord}_v(f_i) = g_{\Delta_{\overline{S}}}(v)$$

edges. By the norm-equivariance property, we can compute $g_{\Delta_{\overline{S}}}(v')$ for all other discrete valuations of $K(S)$, just as in Step (1).

Step 3: In this step we prove

Lemma 3.2.10. $(2g-2)\Delta_{\overline{S}} - \overline{E}_S$ is an effective adelic divisor in $\widehat{\mathrm{Div}}(S/k)$.

Proof. For this we may pass to the analytification via Lemma 3.1.7. Since S is normal, we only need to check that $(2g-2)g_{\Delta_{\overline{S}}} - g_{\overline{E}_S}$ is nonnegative on S^{an}. By density, continuity, and the norm-equivariance property, this can be checked at discrete valuations v of $K(S)$ whose value on a uniformizer is $1/e$. In Steps (1) and (2) we found that at such v, $g_{\Delta_{\overline{S}}}(v) = \delta_v$ and $g_{\overline{E}_S}(v) = \epsilon_v$, so we need to show that $(2g-2)\delta_v \geq \epsilon_v$.

This is done by graph theory. From [Zha10, Equation 4.1.4] we have

$$\epsilon_v = \iint_{\Gamma_v \times \Gamma_v} r(x,y)\delta_{K_{X_v}}(x)\mu(y),$$

where μ is the canonical admissible measure of mass 1 associated to K_{X_v} as defined in [Zha93, Theorem 3.2] and $r(x,y)$ is the *resistance function* between $x, y \in \Gamma_v$ (for a quick reference for all of these graph-theoretic terminologies, see the beginning of [Cin11, Section 4]).

We finish the proof by giving the intuition for why our desired bound is true. For a full proof, see [Yua24, Lemma 3.5], which formalizes the following argument. The idea is that we view the metrized graph Γ_v as a circuit where each edge has is a resistor with resistance 1. Then $r(x,y)$ is the effective resistance between points x and y of this circuit. Using what we remember from high school physics, we expect that $r(x,y)$ is at most the sum of the resistances of the individual resistors.[37] There are δ_v edges by definition of Γ_v, so that

$$\epsilon_v = \iint_{\Gamma_v \times \Gamma_v} r(x,y)\delta_{K_{X_v}}(x)\mu(y) \leq \iint_{\Gamma_v \times \Gamma_v} \delta_v \delta_{K_{X_v}}(x)\mu(y) = (2g-2)\delta_v$$

because the divisor K_{X_v} has degree $2g-2$ by Lemma 2.3.10. □

Summarizing,

Proposition 3.2.11. We have equalities

$$\begin{aligned} \pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a}\rangle &= \overline{\pi}_*\langle \omega_{\overline{X}/\overline{S}}, \omega_{\overline{X}/\overline{S}}\rangle - \mathcal{O}(\overline{E}_S) \\ &= 12\lambda_{\overline{S}} - \mathcal{O}(\Delta_{\overline{S}}) - \mathcal{O}(\overline{E}_S) \\ &= 12\lambda_{\overline{S}} - (2g-1)\mathcal{O}(\Delta_{\overline{S}}) + \mathrm{eff} \end{aligned}$$

in $\widehat{\mathrm{Pic}}(S/k)$.

Step 4: We will globalize de Jong's result from Equation 3.2.5. For this, recall the definitions of the admissible adelic line bundle $\overline{\mathcal{O}}(\Delta)_a$ from Section 2.3.4.[38] It has underlying line bundle $\mathcal{O}(\Delta) \in \mathrm{Pic}(X \times_S X)$, and denoting by (π, π) the structure map $X \times_S X \to S$, we have canonical isomorphisms of underlying line bundles

$$(\pi,\pi)_*\langle \overline{\mathcal{O}}(\Delta), \overline{\mathcal{O}}(\Delta), \overline{\mathcal{O}}(\Delta)\rangle \cong (\pi,\pi)_*\langle \overline{\mathcal{O}}(\Delta), \overline{\mathcal{O}}(-\Delta), \overline{\mathcal{O}}(-\Delta)\rangle \cong \pi_*\langle \Delta^*\overline{\mathcal{O}}(-\Delta), \Delta^*\overline{\mathcal{O}}(-\Delta)\rangle \cong \pi_*\langle \omega_{X/S}, \omega_{X/S}\rangle.$$

For the second isomorphism we use the functoriality of the Deligne pairing as in item (2) of Theorem 2.2.14, taking the canonical section 1 of $\mathcal{O}(\Delta)$. We now play the same game as in Step (1). We have an equality

$$\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a}\rangle - (\pi,\pi)_*\langle \mathcal{O}(\Delta)_a, \mathcal{O}(\Delta)_a, \mathcal{O}(\Delta)_a\rangle =: \mathcal{O}(\overline{\Phi}_S) \tag{3.2.12}$$

for some adelic line bundle $\mathcal{O}(\overline{\Phi}_S)$ with underlying line bundle canonically isomoprhic to $\mathcal{O}_S$. In particular, we take $\overline{\Phi}_S \in \widehat{\mathrm{Div}}(S/k)$ to be $\widehat{\mathrm{div}}(s)$, where s is a rational section of underlying line bundles of projective models of $\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a}\rangle - (\pi,\pi)_*\langle \overline{\mathcal{O}}(\Delta)_a, \overline{\mathcal{O}}(\Delta)_a, \overline{\mathcal{O}}(\Delta)_a\rangle$ corresponding to $1 \in \mathcal{O}_S$ under said isomorphism. Again, $\overline{\Phi}_S$ has underlying divisor 0.

[37] More formally, this is called *Rayleigh's monotonicity law* in circuit theory. The idea is simply that the maximum possible resistance is given by forcing the current to go through all of the resistors in order, without allowing any branches, in which case the entire circuit is in series. This corresponds to the situation in which the metrized graph is just a line.

[38] The notation $\overline{\mathcal{O}}(\Delta)_a$ is rather unfortunate, as here Δ means the diagonal embedding $X \to X \times_S X$ and its associated divisor, as opposed to the very different $\Delta_{\overline{S}}$ from Step (2). On the other hand, we will keep this notation in order to be consistent with [Yua24].

We would like to compute the Green's function of $\overline{\Phi}_S^{an}$, so it suffices to do this at discrete valuations v of $K(S)$. For this, we use the integration formula for the metrics of the Deligne pairing over discretely valued non-Archimedean fields, as in [YZ24, Section 4.6.2]. First, choose a strongly regular sequence of global sections $s_1 = 1, s_2, s_3$ of $\overline{\mathcal{O}}(\Delta)_a^{an}$ on X^{an}.[39] After passing to the base change over $\mathrm{Spec}(\mathcal{O}_{H_v}) \to \overline{S}$, the metric of the Deligne pairing $(\pi,\pi)_*\langle \overline{\mathcal{O}}(\Delta)_a, \overline{\mathcal{O}}(\Delta)_a, \overline{\mathcal{O}}(\Delta)_a\rangle$ at v is then given by

$$-\log \|(\pi,\pi)_*\langle 1, s_2, s_3\rangle\|_v = -\int_{X^{an}_{\mathcal{O}_{H_v}}} \log \|1\|_{\Delta,a,v}\, c_1(\overline{\mathcal{O}}(\Delta)^{an}_{a,v})^2 - \int_{\mathrm{div}(1)^{an}} \log \|s_2\|_{\Delta,a,v}\, c_1(\overline{\mathcal{O}}(\Delta)^{an}_{a,v}) - \int_{(\mathrm{div}(1)\cap \mathrm{div}(s_2))^{an}} \log \|s_3\|_{\Delta,a,v}\,.$$

Here $(\pi,\pi)_*\langle 1, s_2, s_3\rangle$ is a section of the underlying line bundle of $(\pi,\pi)_*\langle \overline{\mathcal{O}}(\Delta)_a, \overline{\mathcal{O}}(\Delta)_a, \overline{\mathcal{O}}(\Delta)_a\rangle$ as defined in [YZ24, Section 4.2.1] (also see item (4) of Theorem 2.2.14), and c_1 denotes the Chambert-Loir measure as explained in [YZ24, Section 3.6.7]. On the other hand, since $\mathrm{div}(1) = \Delta = X$, and if we denote by $s_2^\vee$, $s_3^\vee$ the dual global sections of $\overline{\mathcal{O}}(-\Delta)_a^{an}$,[40] the second and third terms are simply

$$-\int_{X^{an}_{\mathcal{O}_{H_v}}} \log \|s_2^\vee|_X\|_{a,v}\, c_1(\Delta^*\overline{\mathcal{O}}(-\Delta)^{an}_{a,v}) - \int_{\mathrm{div}(s_2^\vee|_X)^{an}} \log \|s_3^\vee|_X\|_{a,v}\,.$$

Since $s_2^\vee|_X = \Delta^* s_2^\vee$ and $s_3^\vee|_X = \Delta^* s_3^\vee$ form a strongly regular sequence of global sections of $\Delta^*\overline{\mathcal{O}}(-\Delta)_a^{an} = \overline{\omega}_{X/S,a}^{an}$ on $X^{an}_{\mathcal{O}_{H_v}}$, the above is equal to $-\log \|\pi_*\langle s_2^\vee|_X, s_3^\vee|_X\rangle\|_v$ which gives the metric of $\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a}\rangle$ at v. Note also that the pullback of the admissible metric on $\overline{\mathcal{O}}(-\Delta)_a$ via Δ is the admissible metric on $\overline{\omega}_{X/S,a}$ by the construction of Theorem 2.3.58.

Therefore if we take our rational section s to be the difference of the sections $\pi_*\langle s_2^\vee|_X, s_3^\vee|_X\rangle$ and $(\pi,\pi)_*\langle 1, s_2, s_3\rangle$, we immediately see that

$$g_{\overline{\Phi}_S}(v) = -\log \|s(v)\| = \int_{X^{an}} \log \|1\|_{\Delta,a,v}\, c_1(\overline{\mathcal{O}}(\Delta)^{an}_{a,v})^2. \tag{3.2.13}$$

By [Zha10, Lemma 3.5.4] the right-hand side is given by

$$-\frac{1}{4}\delta_v(X) - \frac{1}{4}\int_{\Gamma_v} g_\mu(x,x)(K_{X_v} - (10g+2))d\mu =: \varphi_v(X)$$

in the case that $|\pi_v|_v = 1/e$. The notations are as in Step (1) (again μ is the canonical admissible metric on Γ_v associated to K_{X_v}). This is Zhang's *φ-invariant*, which is another graph-theoretic invariant just like ϵ_v and δ_v defined above.

[39] Of course what we really are doing is writing $\overline{\mathcal{O}}(\Delta)_a^{an}$ as a difference of very ample line bundles, and take such a sequence of global sections from those very ample line bundles, but nothing changes in the argument since all the constructions are extended via linearity, and I am too lazy to introduce the notation.

[40] See the previous footnote for the justification of this.

To globalize de Jong's result from Equation 3.2.5, we define an adelic line bundle $\overline{M} \coloneqq 2\overline{\mathcal{O}}(\Delta)_a + p_1^*\overline{\omega}_{X/S,a} + p_2^*\overline{\omega}_{X/S,a}$ on $X \times_S X$, where p_1, p_2 are the two projections to X. It is proved in [Yua24, Theorem 2.10(2)] that $\overline{M}$ is nef, and the key result is

Theorem 3.2.12. [Yua24, Theorem 3.6] There is an equality of adelic line bundles in $\widehat{\mathrm{Pic}}(S/k)_{\mathbf{Q}}$

$$(\pi,\pi)_*\langle \overline{M}, \overline{M}, \overline{M}\rangle = (12g-4)\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a}\rangle - 8\mathcal{O}(\overline{\Phi}_S).$$

To save time we will not prove this theorem here. The idea of the proof, which is not long, is to reduce it to a series of several intermediate identities that compare the Deligne pairings of various line bundles $\overline{\mathcal{O}}(\Delta)_a, p_1^*\overline{\omega}_{X/S,a}, p_2^*\overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a}$. This is done using the functoriality/projection formulas as in [YZ24, Theorem 4.6.1(1)] via factorizing $(\pi,\pi) : X \times_S X \to S$ as a composition $(\pi,\pi) = \pi \circ p_1$ of flat maps.

Therefore we obtain Equation 3.2.7: in $\widehat{\mathrm{Pic}}(S/k)_{\mathbf{Q}}$,

$$\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a}\rangle = \frac{2}{3g-1}\mathcal{O}(\overline{\Phi}_S) + \text{nef}.$$

Note that $(\pi,\pi)_*\langle \overline{M}, \overline{M}, \overline{M}\rangle$ is nef because it is a Deligne pairing of nef adelic line bundles; see Definition-Theorem 2.3.46.

Remark 3.2.13. The results of this step also hold in the arithmetic case $k = \mathbf{Z}$ via a similar integration formula in [YZ24, Section 4.2.2] and [Zha10, Proposition 2.5.3].

Step 5: The last step is to show the following lemma:

Lemma 3.2.14. In $\widehat{\mathrm{Div}}(S)_{\mathbf{Q}}$,

$$\overline{\Phi}_S - \frac{1}{39}\Delta_{\overline{S}}$$

is an effective adelic divisor.

Proof. The proof is the same idea as in Lemma 3.2.10. Passing to the analytification by Lemma 3.1.7, we just need to check that $g_{\overline{\Phi}_S} - g_{\Delta_{\overline{S}}}/39$ is nonnegative, which can be done at discrete valuations v of $K(S)$ with $|\pi_v|_v = 1/e$ by the norm-equivariance. In that case this quantity is just $\varphi_v - \delta_v/39$, which is nonnegative as proved by [Cin11, Theorem 2.11]. □

To conclude once more, combine Lemma 3.2.14 with the conclusion of Step (4) to obtain Equation 3.2.9,

$$\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a}\rangle = \frac{2}{39(3g-1)}\mathcal{O}(\Delta_{\overline{S}}) + \text{nef} + \text{eff}.$$

Combine $39(3g-1)(2g-1)/2$ times the above equation with the equation

$$\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a}\rangle = 12\lambda_{\overline{S}} - (2g-1)\mathcal{O}(\Delta_{\overline{S}}) + \text{eff}$$

from Proposition 3.2.11, and we get

$$\left(1+\frac{39(3g-1)(2g-1)}{2}\right)\pi_*\langle\overline{\omega}_{X/S,a},\overline{\omega}_{X/S,a}\rangle = 12\lambda_{\overline{S}} + \text{nef} + \text{eff},$$

and the proof is complete.

Remark 3.2.15. It might be worthwhile to step back and take a look at what just happened. Indeed, the inputs we needed for the actual *computations* all came from previous work by Zhang, Cinkir, etc. The only use of the adelic line bundles was to provide the language to assemble these computations into a global/relative and geometric "package," which will then be put to great use later in Section 3.3. In this sense the theory of adelic line bundles is much like the the theory of schemes in that schemes allow one to do a similar globalization/relativization of classical facts involving classical varieties over fields (via functor of points, etc.).[41]

3.2.3 A "Gross–Schoen" line bundle

In this section we work over number fields (so $k = \mathbf{Z}$) in order to match the discussion from the introduction and the setting of [GZ24], but many of the constructions and results also hold for function fields of one variable. Once again the genus of all (relative) curves will satsify $g > 1$.

Recall from Definition 1.1.3 that for point $[C] \in \mathcal{M}_g(\mathbf{Q})$, we may define a real number $h_{GS}([C]) := \langle GS(C), GS(C)\rangle_{BB}$ which is the Beilinson–Bloch height of the Gross–Schoen cycle of the curve C. The latter notion was defined in Definition 1.1.2; we recall that the definition of $GS(C)$ requires the choice of a divisor $e_{can} \in \mathrm{Div}(C)$ satisfying $(2g-2)e_{can} \sim \omega_C$, which is fixed once and for all.

The main result of [Zha10] is:

Theorem 3.2.16. [Zha10, Corollary 1.3.2] For a smooth projective curve C defined over a number field K,

$$h_{GS}(C) = \frac{2g+1}{2g-2}\overline{\omega}^2_{C/K,a} - \sum_{v\in M_K}\varphi(C_v)\log(n_v).$$

By our work in Section 3.2.2 this equality can be globalized. Namely, let S be a quasiprojective normal variety over K, $\pi : X \to S$ a smooth relative curve of genus $g > 1$. Define $\overline{L} \in \widehat{\mathrm{Pic}}(S/k)_{\mathbf{Q}}$ as

$$\overline{L} := \frac{2g+1}{2g-2}\pi_*\langle\overline{\omega}_{X/S,a},\overline{\omega}_{X/S,a}\rangle - \mathcal{O}(\overline{\Phi}_S).$$

[41] Disclaimer: this is only my personal intuition as a student.

The importance of this (integrable) adelic $\mathbf{Q}$-line bundle $\overline{L}$ is that $h_{\overline{L}}(s) = h_{GS}(X_s)$ for all $x \in S(\overline{K})$; we will use this later after we see its positivity properties.

In [Yua24, Section 3.3.5] Yuan asked the question of whether or not $\overline{L}$ is nef and big on S, assuming $\pi : X \to S$ has maximal variation. Of course the issue is to control the $\mathcal{O}(\overline{\Phi}_S)$ term. This question is partially answered by Gao–Zhang by [GZ24, Theorem 6.5]. Below we will state their result and give some indications of the proof, which we unfortunately do not have time to give details of. Broadly speaking, their method is geometric and Hodge-theoretic in nature rather than a globalization of "local" results obtained from combinatorial methods, as in the proof of Theorem 3.2.8.

Remark 3.2.17. An important remark is that if C is hyperelliptic, [Zha10, Corollary 1.3.3] shows that $\frac{2g+1}{2g-2}\overline{\omega}^2_{C/K,a} = \sum_{v \in M_K} \varphi(C_v) \log(n_v)$. Hence we may assume from now on that $g \geq 3$, because $h_{GS}(C) = 0$ is not interesting when $g = 2$.

First, the line bundle $\overline{L}$ is constructed in a different manner. Rather than the explicit description above, consider a fine moduli space $\mathcal{M}_g$ of genus-g curves over $\mathbf{Q}$, where we have imposed some level structure that is omitted from the notation. Let $\pi_g : \mathcal{C}_g \to \mathcal{M}_g$ be the universal curve, and $\mathcal{J}_g \coloneqq \mathrm{Jac}(\mathcal{C}_g/\mathcal{M}_g)$ be the relative Jacobian. Then the Poincaré line bundle $\mathcal{P}$ on $\mathcal{J}_g \times_{\mathcal{M}_g} \mathcal{J}_g$ extends to an integrable adelic $\mathbf{Q}$-line bundle $\overline{\mathcal{P}} \in \widehat{\mathrm{Pic}}(\mathcal{J}_g \times_{\mathcal{M}_g} \mathcal{J}_g/\mathbf{Z})_{\mathbf{Q}}$ by Tate's limit process for adelic line bundles as outlined in [YZ24, Section 6.1.1]. Now, choose a canonical class $\xi \in \mathrm{Pic}^1(\mathcal{C}_g/\mathcal{M}_g)$ such that $(2g-2)\xi = \omega_{\mathcal{C}_g/\mathcal{M}_g}$, and let $i : \mathcal{C}_g \to \mathcal{J}_g$ be the Abel–Jacobi map based at ξ. Pulling back $\overline{\mathcal{P}}$ via $(i, i) : \mathcal{C}_g \times_{\mathcal{M}_g} \mathcal{C}_g \to \mathcal{J}_g \times_{\mathcal{M}_g} \mathcal{J}_g$, we get an integrable adelic line bundle $\overline{\mathcal{Q}} \in \widehat{\mathrm{Pic}}(\mathcal{C}_g \times_{\mathcal{M}_g} \mathcal{C}_g/\mathbf{Z})_{\mathbf{Q}}$. Finally, we take the Deligne pairing $(\pi_g, \pi_g)_*\langle \overline{\mathcal{Q}}, \overline{\mathcal{Q}}, \overline{\mathcal{Q}} \rangle \in \widehat{\mathrm{Pic}}(\mathcal{M}_g/\mathbf{Z})_{\mathbf{Q}}$ via the structure map $(\pi_g, \pi_g) : \mathcal{C}_g \times_{\mathcal{M}_g} \mathcal{C}_g \to \mathcal{M}_g$ of relative dimension 2. It turns out that

Proposition 3.2.18. [GZ24, Theorem 6.1]

$$(\pi_g, \pi_g)_*\langle \overline{\mathcal{Q}}, \overline{\mathcal{Q}}, \overline{\mathcal{Q}} \rangle \cong \overline{L}.$$

Having introduced Gao–Zhang's alternate construction of $\overline{L}$, we now move on to their main results. In fact they do *not* prove that $\overline{L}$ is big as an adelic line bundle on $\mathcal{M}_g$ over $\mathbf{Z}$, but rather that the *generic fiber* $\widetilde{L} \in \widehat{\mathrm{Pic}}(\mathcal{M}_g/\mathbf{Q})_{\mathbf{Q}}$ is big. This will ultimately only give a weaker height bound—compare the forms of the inequalities in Theorem 1.1.4 and 1.1.1, noting in particular the extra constant c appearing in Theorem 1.1.4. Now, to prove this statement, one would in theory want to prove the *nefness* of $\widetilde{L}$ first, so that we may apply the arithmetic Hilbert–Samuel formula that expresses $\widehat{\mathrm{vol}}(\mathcal{M}_g, \widetilde{L})$ as a self-intersection number. This could not be done in their work, but [GZ24, Theorem 6.3] still manages to prove

Theorem 3.2.19. Let $S \subseteq \mathcal{M}_{g,\mathbf{C}}$ be an irreducible subvariety (not necessarily closed), and let $\widetilde{L}_{\mathbf{C}}|_S$ be the image of $\widetilde{L}$ under the natural base change map $\widehat{\mathrm{Pic}}(\mathcal{M}_g/\mathbf{Q})_{\mathbf{Q}} \to \widehat{\mathrm{Pic}}(S/\mathbf{C})_{\mathbf{Q}}$

(see Example 2.3.42 for a quick refresher). Then

$$\widehat{\mathrm{vol}}(S, \widetilde{L}_{\mathbf{C}}|_S) = \int_S c_1(\overline{L}_{\mathbf{C}})^{\wedge \dim(S)}.$$

Here by $c_1(\overline{L}_{\mathbf{C}})$ we mean the limit of (1,1)-forms $c_1(\mathcal{L}_i)$, where $\{\mathcal{L}_i\}$ is a sequence of Hermitian line bundles on projective models $\mathscr{M}_i/\mathbf{C}$ that approximate $\overline{L}_{\mathbf{C}}$. This is the same as the curvature of the analytification of $\overline{L}$ on S^{an} upon restricting to the complex fiber, which becomes a Hermitian line bundle over a complex variety.

The proof of Theorem 3.2.19 relies on constructing a specific sequence of Hermitian line bundles $\mathcal{L}_i$, coming from the "geometric" construction of $\overline{L}$ outlined above, that approximate $\overline{L}_{\mathbf{C}}$. The volumes of these $\mathcal{L}_i$ are then related to the $\int_S c_1(\mathcal{L}_{i,\mathbf{C}})^{\wedge \dim(S)}$ using *Demailly's Morse inequality*, and the passage to the limit is justified by Definition-Theorem 3.1.8 (for the volume) and the proof of [YZ24, Lemma 5.4.4] (for the integral). The details can be found in [GZ24, Section 6.4].

We need one additional fact. It turns out that $c_1(\overline{L}_{\mathbf{C}})$ is equal on $\mathcal{M}_{g,\mathbf{C}}$ to a certain *Betti form* β_{GS} which is defined Hodge-theoretically. It is also known to be semipositive. The much more involved statement that is proved in [GZ24, Corollary 5.3] is that if $g \geq 3$, then $\beta_{GS}^{\wedge(3g-3)} = \beta_{GS}^{\dim(\mathcal{M}_g)}$ is not identically 0. This is the culmination of the geometric part of their argument, which takes up the bulk of their paper.

With these facts in hand, when $S = \mathcal{M}_{g,\mathbf{C}}$ in the setting of Theorem 3.2.19, we get

Theorem 3.2.20. If $g \geq 3$, then $\widetilde{L}$ is big on $\mathcal{M}_g/\mathbf{Q}$.

Proof. Because we are working in the geometric setting and there are no integral models to deal with, we have $\widehat{\mathrm{vol}}(\mathcal{M}_g, \widetilde{L}) = \widehat{\mathrm{vol}}(\mathcal{M}_{g,\mathbf{C}}, \widetilde{L}_{\mathbf{C}})$ because the volumes can be computed by a limit of line bundles on projective models, and the flat base change from $\mathbf{Q}$ to $\mathbf{C}$ does not change any of the h^0's in question. Then we have

$$\widehat{\mathrm{vol}}(\mathcal{M}_g, \widetilde{L}) = \int_{\mathcal{M}_{g,\mathbf{C}}} c_1(\overline{L}_{\mathbf{C}})^{\wedge(3g-3)}$$

and $c_1(\overline{L}_{\mathbf{C}})^{\wedge(3g-3)} = \beta_{GS}^{\wedge(3g-3)}$ is everywhere nonnegative by semipositivity of β_{GS},[42] and also not 0 everywhere. Therefore the above integral is positive. ☐

3.3 Height consequences

Having done the difficult work of proving the bigness of various adelic line bundles, is now time to apply the height inequality, Theorem 3.1.15.

[42] This is easy to check from the definition of a semipositive $(1,1)$ form as one which can locally be written $i\sum_{i,j} h_{ij}(z)dz_i \wedge d\overline{z}_j$ with $(h_{ij}(z))$ a positive semidefinite real symmetric matrix at all z.

3.3.1 Proof of Theorem 1.1.1 in the function field case

Let k be a field and K a function field of one variable over k. Let $X \to S$ be a smooth relative curve of genus $g > 1$ of maximal variation, where S is a quasiprojective normal integral scheme over k. In Theorem 3.2.7 we proved that $\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a}\rangle$ is nef and big on S/k. In particular, take S to be a fine moduli space $\mathcal{M}_{g,k}$, where we have imposed some level structure that is omitted from the notation, and $\pi : X \to S$ the universal curve. In particular S is smooth, so Theorem 3.2.7 applies. Moreover π has a stable compactification $\overline{\pi} : \overline{X} \to \overline{S}$.

Therefore we have nef and big adelic line bundles $\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a}\rangle$ and $\lambda_{\overline{S}}$ on S/k, and pulling back via $f : S_K = \mathcal{M}_{g,K} \to S$, $f^*(\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a}\rangle) \cong \pi_*\langle \overline{\omega}_{X_K/S_K,a}, \overline{\omega}_{X_K/S_K,a}\rangle$ (functoriality of the Deligne pairing, [YZ24, Theorem 4.1.3]) and $f^*\lambda_{\overline{S}} \cong \lambda_{\overline{S}_K}$ are nef adelic line bundles in $\widehat{\mathrm{Pic}}(S_K/k)$. Now, if we pick a quasiprojective normal model $\mathcal{U}$ of S_K over k in which the rational map $h : \mathcal{U} \to S$ extending f is fully defined, then $\pi_*\langle \overline{\omega}_{X_K/S_K,a}, \overline{\omega}_{X_K/S_K,a}\rangle$ is by definition represented by $h^*\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a}\rangle \in \widehat{\mathrm{Pic}}(\mathcal{U}/k)$, which is big by Theorem 3.2.7. Since effectivity, hence volumes and bigness, can be checked on quasiprojective models (see Definition 3.1.5), we conclude that $\pi_*\langle \overline{\omega}_{X_K/S_K,a}, \overline{\omega}_{X_K/S_K,a}\rangle \in \widehat{\mathrm{Pic}}(S_K/k)$ is big.

Also, by Theorem 3.2.5, the generic fiber of $\lambda_{\overline{S}_K}$ is big in $\widehat{\mathrm{Pic}}(S_K/K)$. By parts (1) and (3) of the height inequality Theorem 3.1.15 applied to the identity map $S_K \to S_K$, we can thus find nonempty open subvariety U_K of S_K and constants $c_1, c_2, d > 0$ only depending on g[43] such that

$$c_1 h_{\lambda_{\overline{S}_K}} \le h_{\pi_*\langle \overline{\omega}_{X_K/S_K,a}, \overline{\omega}_{X_K/S_K,a}\rangle} \le c_2 h_{\lambda_{\overline{S}_K}} + d \tag{3.3.1}$$

as height functions on $U_K(\overline{K}) = U(\overline{K})$. We can restrict $\pi_*\langle \overline{\omega}_{X_K/S_K,a}, \overline{\omega}_{X_K/S_K,a}\rangle$ and $\lambda_{\overline{S}_K}$ to $S_K - U_K$, where the exact same facts imply that the former is still big over k and the latter is still big generically (over K), and produce an inequality of the above form on a dense open subvariety of $S_K - U_K$. So by induction on dimension we have such an inequality valid on all of $S_K(\overline{K}) = S(\overline{K})$.

From the definitions we have

$$h_{\lambda_{\overline{S}}}(s) = h_{\lambda_{\overline{S}_K}} = h_{Fal}(X_s),$$

which is Example 3.2.6 (where we write $h_{\lambda_{\overline{S}}}(s)$ for $s \in S(\overline{K})$ by abuse of notation), and similarly

$$h_{\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a}\rangle}(s) = h_{\pi_*\langle \overline{\omega}_{X_K/S_K,a}, \overline{\omega}_{X_K/S_K,a}\rangle} = \overline{\omega}^2_{X_s/K,a}.$$

[43] We will omit the details, but c_1, c_2, d do not depend on K because we are really calculating the heights of (1-dimensional) points $s \in S(K)$, so the base change to S_K is superfluous (except for setting up things in the framework of Definition 3.1.1) and the relevant constants only depend on S/k anyways. There is no dependency on k since the relevant volumes (dimensions of vector spaces) in the key input to the height inequality, which is Theorem 3.1.13, obviously do not depend on k.

Therefore Theorem 1.1.1 is obtained upon noting that in the non-isotrivial case, $h_{Fal}(X_s) > 4^{-4g^2}$ [Yua24, Lemma 4.9]. This allows up to replace the first inequality $c_1 h_{Fal}(X_s) \leq \overline{\omega}^2_{X_s/K,a}$ in (3.3.1) with $c_1 \max(h_{Fal}(X_s), 1) \leq \overline{\omega}^2_{X_s/K,a}$ if we allow ourselves to replace c_1 by a smaller positive constant, still only depending on g. Also, we have $d \leq c_3(h_{Fal}(X_s))$ for some constant $c_3 > 0$, so upon replacing c_2 by $c_2 + c_3$ and then $h_{Fal}(X_s)$ by the larger quantity $\max(h_{Fal}(X_s), 1)$, we have

$$c_1 \max(h_{Fal}(X_s), 1) \leq \overline{\omega}^2_{X_s/K,a} \leq c_2 \max(h_{Fal}(X_s), 1). \tag{3.3.2}$$

Remark 3.3.1. The above proof does not give explicit values for c_1 and c_2. On the other hand, it is possible to explicitly obtain $c_1 = 1/12$ and $c_2 = 12$ as in [Yua24, Theorem 4.11]. The interesting thing about the explicit proof is that it completely bypasses the "uniform"/relative nature of the above proof, and instead verifies the inequalities on single (arbitrary) curves. On the other hand, in the *number field case*, which we did not do, this explicit "single-curve proof" is not possible, and instead one has to use the (arithmetic analogues of the) various adelic line bundles/divisors defined above to get the (implicitly defined) constants.

Also note that any improvement on the constant 12 in the upper bound essentially implies the effective Mordell conjecture, as mentioned in the introduction.

In fact we have essentially enough setup to be able to give the explicit values for c_1 and c_2. We will prove the following explicit version of Theorem 1.1.1 (still in the function field case):

Theorem 3.3.2 (Theorem 4.11, [Yua24])**.** Let K be a function field of one variable over a field k. Let C be a geometrically integral smooth projective curve of genus $g > 1$ over K. Then

$$\frac{1}{12} h_{Fal}(C) \leq \overline{\omega}^2_{C/K,a} \leq 12 h_{Fal}(C).$$

In particular, by [Yua24, Lemma 4.9], in the case that $C_{\overline{K}}$ is non-isotrivial over $\overline{k}$, we have the better bound

$$\frac{1}{12} \max(h_{Fal}(C), 4^{-4g^2}) \leq \overline{\omega}^2_{C/K,a} \leq 12 \max(h_{Fal}(C), 1).$$

Proof. Since both the stable Faltings height and the self-intersection of the admissible dualizing sheaf are unaffected by finite extension of the base field K, we may assume that C has a semistable *regular* model $\mathcal{C}$ over S, where S is the unique smooth projective k-curve with function field K. Recall from the proof of Theorem 3.2.8 in Section 3.2.2 that there are graph-theoretic invariants $\epsilon_v, \delta_v, \varphi_v$ of the fiber $\mathcal{C}_v$ over all places v of K (i.e. closed points v of S), which were defined in Steps 1, 2, and 4 of the proof respectively. We define $\epsilon(C) = \sum_v \epsilon_v(C) \log(n_v)$, where $n_v = \#k(v)$, and likewise for $\delta(C)$ and $\varphi(C)$.

Take (Arakelov) degrees in the equality of metrized line bundles given in Equation 3.2.2, and we find that $\overline{\omega}^2_{C/K,a} = \omega^2_{\mathcal{C}/S} - \epsilon(C)$. From the Noether formula (3.2.11) we can also take degrees and obtain $12h_{Fal}(C) = \omega^2_{\mathcal{C}/S} + \delta(C)$, and so

$$\overline{\omega}^2_{C/K,a} = 12h_{Fal}(C) - \delta(C) - \epsilon(C). \tag{3.3.3}$$

Since the local invariants δ_v, ϵ_v are all nonnegative, this gives the $\overline{\omega}^2_{C/K,a} \leq 12h_{Fal}(C)$ part of the theorem.

Next, [dJ18, Proposition 9.2] gives the equality

$$\delta + \epsilon - 2\varphi = \frac{3}{2}\varphi - 6\tau \leq \frac{3}{2}\varphi$$

for invariants of polarized metrized graphs, where the τ-invariant is another invariant of polarized metrized graphs defined via integrations of Green's functions (just like ϵ and τ). We will refer to [dJ18, Equation 1.9] and the discussions in [Cin11, pgs. 530–531] for more information about the τ constant. It is nonnegative for any polarized metrized graph by [Cin11, Lemma 3.6], which justfies the inequality.

Now, recall that [Cin11, Theorem 2.11] proved that $\varphi(C) \geq c(g)\delta(C)$ for some function $c(g)$ of g, which satisfies $c(2) = 1/27$, $c(3) = 17/288$,[44] $c(4) = 3/88$, and $c(g) \geq 1/25$ for all $g \geq 5$. Then we have

$$\delta(C) + \epsilon(C) \leq \left(\frac{3}{2c(g)} + 2\right)\varphi(C),$$

Combine the above equation with (3.3.3) to get

$$\overline{\omega}^2_{C/K,a} \geq 12h_{Fal}(C) - \left(\frac{3}{2c(g)} + 2\right)\varphi(C) \tag{3.3.4}$$

We also have

$$\overline{\omega}^2_{C/K,a} \geq \frac{g-1}{2g+1}\varphi(C)$$

from [LSW21, Proposition 6.1], and combining this with (3.3.4) we get

$$\left(1 + \left(\frac{3}{2c(g)} + 2\right)\frac{2g+1}{g-1}\right)\overline{\omega}^2_{C/K,a} \geq 12h_{Fal}(C). \tag{3.3.5}$$

For $g \geq 3$, the coefficient in front of $\overline{\omega}^2_{C/K,a}$ may be calculated to be less than 139 in all cases, which gives the first inequality of the theorem, $h_{Fal}(C)/12 \leq \overline{\omega}^2_{C/K,a}$. In the $g = 2$ case (in fact for any hyperelliptic C), [Zha10, Corollary 1.3.3] gives

$$\overline{\omega}^2_{C/K,a} = \frac{2g-2}{2g+1}\varphi(C),$$

[44] Note that the constant 17/288 is actually a product of Cinkir's later work [Cin15] that refines the graph-theoretic calculations in the genus 3 case via explicit casework.

so that the inequality (3.3.5) holds with the $g-1$ in the denominator of the coefficient in the left-hand side replaced with $2g-2$. This new coefficient can be estimated to be less than 108 in the case $c(2) = 1/27$, so we again have the desired inequality $h_{Fal}(C)/12 \le \overline{\omega}^2_{C/K,a}$. $\square$

3.3.2 Proof of Theorem 1.1.4

In this section, we continue the notation of Section 3.2.3; in particular we are in the arithmetic case $k = \mathbf{Z}$.

To prove Theorem 1.1.4, we first make a reduction. As explained before the statement of the theorem, we only need to take care of the inequalities for h_{GS} as it is equal to $6h_{Ce}$. Moreover, [Zha10, Corollary 2.5.2] shows that for a smooth projective curve C (over a number field K) and $e \in \mathrm{Pic}^1(C)$,

$$\langle GS_e(C), GS_e(C)\rangle_{BB} = \langle GS(C), GS(C)\rangle_{BB} + 6(g-1)h_{NT}(e - e_{can}).$$

Therefore to prove that the inequality

$$\langle GS_e(\mathcal{C}_s), GS_e(\mathcal{C}_s)\rangle_{BB} \ge \epsilon(h_{Fal}(\mathcal{C}_s) + h_{NT}(e - e_{can,s})) - c$$

holds for all $\overline{\mathbf{Q}}$-points s in some open dense subscheme U of $\mathcal{M}_g$ and all $e \in \mathrm{Pic}^1(\mathcal{C}_s)$, we only have to prove

$$h_{GS}(s) = \langle GS(\mathcal{C}_s), GS(\mathcal{C}_s)\rangle_{BB} \ge \epsilon h_{Fal}(\mathcal{C}_s) - c. \tag{3.3.6}$$

In Theorem 3.2.20 we proved that the generic fiber $\widetilde{L} \in \mathrm{Pic}(\mathcal{M}_g/\mathbf{Q})$ of $\overline{L} \in \mathrm{Pic}(\mathcal{M}_g/\mathbf{Z})$ is big. Then we apply part (3) of the height inequality Theorem 3.1.15 to the identity morphism $\mathcal{M}_g \to \mathcal{M}_g$ over $\mathbf{Q}$, and the role of $\overline{M}$ in that theorem will be played by the Hodge bundle $\overline{\lambda}_{\mathcal{M}_g}$ as constructed in Theorem 2.3.3, whose associated height is exactly $h_{Fal}(\mathcal{C}_s)$. This gives (3.3.6).

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